ON $\mathscr{H}_\infty$-BASED DIRECTION FINDING AND SOURCE DETECTION

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ABSTRACT

Recently, the authors have proposed a novel approach to Direction-Of-Arrival (DOA) estimation using the $\mathscr{H}_\infty$ criterion. This article examines the limits of performance of $\mathscr{H}_\infty$-based direction finding and arrives at a lower bound on the estimation error, analogous to the Cramer-Rao Bound (CRB). Furthermore, the analysis of the $\mathscr{H}_\infty$-based algorithm's prediction error using one of Akaike's information criteria shows that the direction finding algorithm at hand possesses source detection ability.

1. INTRODUCTION

In DOA estimation, small deviations from the Gaussian assumptions about the "noise"—which comprises all that is unmodeled of the signal—may lead to large errors in the angles of arrival. The pertinent literature has tackled the issue, but most attempts have stayed within the probabilistic framework [1, 2]. The authors have addressed the problem by treating the "noise" as a worst-case disturbance and proposing an $\mathscr{H}_\infty$-based DOA estimator [3, 4, 5, 6]. In the present article, the authors investigate the efficiency and detection ability of $\mathscr{H}_\infty$-based direction finding. Some works on the CRB of deterministic systems exist in the literature [7, 8]. At the same time, results on the use of information criteria in the model selection problem for sinusoidal signals have been reported [9, 10].

In recent work [4], the authors have expressed the response of a Uniform Linear Array (ULA) of J sensors to p narrowband far-field sources in the form of the following state equation:

$$\begin{aligned} z_{l+1} &= f^p(z_l),\ z_l \in \mathbf{C}^{2p} \\ y_l &= h^p(z_l) + w_l \\ \theta &= L z_l,\ L = [0_p\, I_p] \end{aligned} \tag{1.1}$$

where $\theta \in \mathbf{R}^p$ is the vector of the azimuth angles of arrival, $y_l \in \mathbf{C}$ is the measured signal at the l–th sensor, and $w_l \in \mathbf{C}$ is the uncertainty about the array model (e.g., amplitude and phase variations). Herein, the uncertainty is deterministic and the aim of the $\mathscr{H}_\infty$-based method is to attain the best possible performance against the worst of all possible disturbances, w_l. The nonlinear mappings f^p and h^p are given in the Appendix—the superscript indicates that they depend on the number of sources, p. The only assumption made on the disturbance, w, is that $\|w\|^2_{2,[0,J]} < \infty$, where

$$\|v\|^2_{2,[0,J]} := \sum_{l=0}^{l=J} \|v_l\|_2^2$$

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The authors' DOA estimator is a single-snapshot algorithm in the form of the state-space filter [4]:

$$\begin{aligned} \hat{z}_{l+1} &= f^p(\hat{z}_l) + N_l[y_l - h^p(\hat{z}_l)] \\ \hat{\theta}_l &= L\hat{z}_l \end{aligned} \tag{1.2}$$

where $N_l \in \mathbf{C}^{2p}$ is solved for in accordance with the $\mathscr{H}_\infty$ criterion. From (1.1) and (1.2), one easily gets the errors in the state, e_z, and DOA estimates, e_θ, as follows:

$$\begin{aligned} e_{z,l+1} &= f^p(z_l) - f^p(\hat{z}_l) - N_l[h^p(z_l) - h^p(\hat{z}_l) + w_l] \\ e_{\theta,l} &= L e_{z,l} \end{aligned} \tag{1.3}$$

In the $\mathscr{H}_\infty$ framework, one computes N_l in (1.2) to minimize the effect of the worst-case disturbance on the DOA estimation error. Equivalently, the filter's gain stems from the solution of the following min-max problem:

$$\inf_N \sup_w \frac{\|e_\theta\|^2_{2,[0,J]}}{\|w\|^2_{2,[0,J]} + e^*_{z,0} R e_{z,0}} = \gamma^2_{\inf} \tag{1.4}$$

where the tolerance, $\gamma_{\inf}$, is a constant whose—generally unknown—value is a property of (1.3). The positive definite matrix R measures the significance of the initial state estimation error and * denotes the complex conjugate transpose.

First, the present article investigates the efficiency of the $\mathscr{H}_\infty$-based algorithm. Because the algorithm follows a worst-case rather than a probabilistic approach, one is unable to employ the CRB. Instead, we derive an analogous lower bound as well as an upper bound on the DOA estimation error. Second, we show that the $\mathscr{H}_\infty$-based DOA estimator can be used for source detection. The order of the state equation (1.1) is directly proportional to the number of sources. Analysis of the prediction error using information criteria reveals the order of the equation hence the number of sources that best fit the data.

2. EFFICIENCY OF $\mathscr{H}_\infty$-BASED DOA ESTIMATION

As mentioned earlier, the smallest possible value of the min-max problem, $\gamma_{\inf}$, is unknown in general. Then, one proceeds by using $\gamma \geq \gamma_{\inf}$ in the right-hand-side of (1.4) as follows:

$$\inf_N \sup_w \frac{\|e_\theta\|^2_{2,[0,J]}}{\|w\|^2_{2,[0,J]} + e^*_{z,0} R e_{z,0}} = \gamma^2$$

Evaluating the quotient at the saddle point $(w^\#, N^\#)$ of the above min-max problem, we have

$$\|e^\#_\theta\|^2_{2,[0,J]} \geq \gamma^2_{\inf} \left[\|w^\#\|^2_{2,[0,J]} + e^*_{z,0} R e_{z,0} \right]$$

In words, the worst-case estimation error is bounded below by the respective disturbance (plus the initial state estimation error) times the minimum tolerance. A similar result for (deterministic) ARMA models has been reported in the literature [8]. Unlike the CRB, which uses the likelihood function, the aforementioned lower bound is associated with the worst-case performance of the estimator. As with the CRB, however, a priori computation of the lower bound is impossible.

An advantage of the $\mathscr{H}_\infty$-based approach is that it naturally provides an upper bound on the DOA estimation error. For any disturbance, w, we have

$$\|e_\theta\|^2_{2,[0,J]} \leq \gamma^2 \left[\|w\|^2_{2,[0,J]} + e^*_{z,0} R e_{z,0}\right] \tag{2.5}$$

From [3], the DOA estimation error converges and, therefore, for some l_0 it follows that

$$\left\|e_{\theta,l}\right\|_2^2 \geq \left\|e_{\theta,J}\right\|_2^2, \; l \geq l_0 + 1 \tag{2.6}$$

By definition,

$$\|e_\theta\|^2_{2,[0,J]} = \|e_\theta\|^2_{2,[0,l_0]} + \|e_\theta\|^2_{2,[l_0+1,J]} \tag{2.7}$$

Using (2.6) and (2.7), inequality (2.5) leads to

$$\|e_\theta\|^2_{2,[0,l_0]} + (J - l_0) \left\|e_{\theta,J}\right\|_2^2 \leq \gamma^2 \left[\|w\|^2_{2,[0,J]} + e^*_{z,0} R e_{z,0}\right]$$

In turn, the above yields the following upper bound on the final DOA estimation error

$$\left\|e_{\theta,J}\right\|_2^2 \leq \frac{\gamma^2}{J - l_0} \left[\|w\|^2_{2,[0,J]} + e^*_{z,0} R e_{z,0}\right]$$

Clearly, as the number of sensors, J, increases the DOA estimation error gets smaller.

3. SOURCE DETECTION USING $\mathscr{H}_\infty$-BASED DIRECTION FINDING

The proposed source detection algorithm uses the Final Prediction Error (FPE) of the $\mathscr{H}_\infty$-based DOA estimator. The assumed number of signals, p, ranges over the set of all possible sources; thus, $p \in \{0, 1, \ldots, J-1\}$ where J is the number of sensors.

First, one executes the direction finding algorithm (1.2) for $p = 0, 1, \ldots, J-1$ and records the resulting vector of DOA estimates, $\hat{\theta}^p_J$. Notice that (1.2) is recursive along the array aperture; thus, $\hat{\theta}^p_J$ is obtained after the J-th sensor in a single-snapshot is processed. Second, based on (1.1) consider the predictor model below:

$$\begin{aligned} \bar{z}^p_{l+1} &= f^p(\bar{z}^p_l) \\ \bar{y}^p_l &= h^p(\bar{z}^p_l) \end{aligned} \tag{3.8}$$

To generate the predicted output, $\bar{y}^p_l$, one uses the final DOA estimate, $\hat{\theta}^p_J$, in (3.8); see Appendix. At this point, we compute the FPE as follows:

$$\varepsilon_{lp} = y_l - \bar{y}^p_l$$

Third, we employ Akaike's FPE criterion [11]

$$\Psi_p = \frac{1 + \frac{d_p}{J}}{1 - \frac{d_p}{J}} \frac{1}{J} \sum_{l=0}^{l=J} \frac{1}{2} \varepsilon^2_{lp} \tag{3.9}$$

and the number of sources is equal to the value of p that yields the minimum Ψ_p. The number of degrees of freedom in the array model (1.1) is d_p. In the particular case where the only unknown parameter per source in the array model is azimuth, one has $d_p = p$. The authors investigated other criteria, such as the Minimum Description Length, but Akaike's FPE emerged as the most suitable for the approach of this paper.

3.1 Numerical Example

Consider a situation with two sources in the far field and 10 dB Signal-to-Noise Ratio (SNR). Using a ULA comprising $J = 25$ sensors and a single snapshot of data per trial, we performed one hundred Monte-Carlo trials for each assumed number of sources. Without loss of generality, we limited the possible number of signals as follows: $p \in \{0, 1, 2\}$. Fig. 1 summarizes the results and shows 76% success rate.

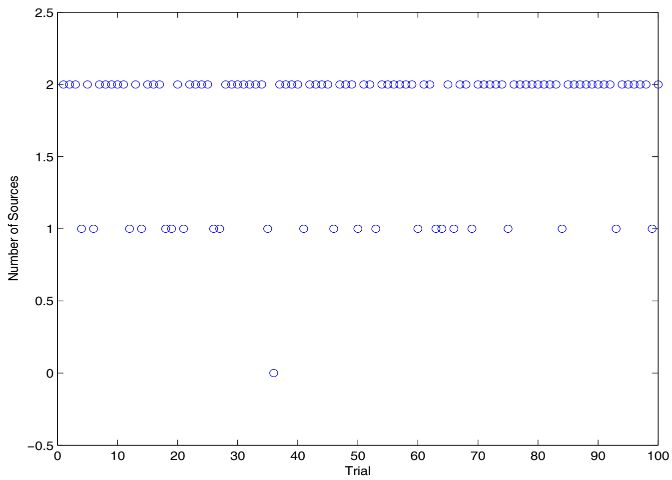

Figure 1: Source detection using a single snapshot; two sources present

Next consider one source present in the far field at 10 dB SNR. The results from the Monte-Carlo trials of FPE criterion (3.9), depicted in Fig 2, indicate 67% success rate. Last, the detection algorithm was successful 100% of the time when no source was present in the far field.

4. CONCLUSION

The investigation of the $\mathscr{H}_\infty$-based DOA estimator's efficiency has yielded CRB-like (lower) bound as well as an upper bound. Moreover, analysis of the $\mathscr{H}_\infty$ filter's prediction error using Akaike's FPE criterion shows that the algorithm can detect the number of sources in the far field.

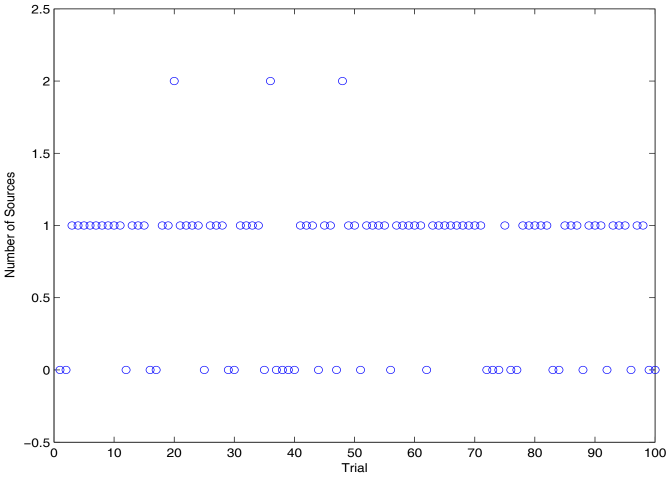

Figure 2: Source detection using a single snapshot; one source present.

A. APPENDIX

The definitions of the nonlinear mappings f^p and g^p are given below:

$$f^p(z_l) := \begin{bmatrix} A(\theta)x_l \\ \theta \end{bmatrix}$$
$$h^p(z_l) := C(\theta)x_l$$

where $A(\theta)$ is a diagonal matrix with entries

$$[A(\theta)]_i = \exp\left[-j\omega\tau(\theta_i)\right], \; i = 1, \cdots, p$$

and, for unity-magnitude source signals,

$$[C(\theta)]_i = 1, \; i = 1, \cdots, p$$

In the above expressions, ω is the center frequency—common for all signals—and $\tau(\theta_i)$ is the inter-sensor delay associated with the i-th source. Moreover,

$$z_l := \begin{bmatrix} x_l \\ \theta \end{bmatrix}$$

and the components of x_l

$$x_{i,l} := \exp\left[-j\omega\tau(\theta_i)l\right], \; i = 1, \cdots, p$$

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