QUADRATIC WEIGHTED MEDIAN FILTERS FOR NOISY IMAGE SHARPENING

Tuncer Can Aysal and Kenneth E. Barner

Department of Electrical and Computer Engineering, University of Delaware, Newark, DE, USA, 19716
{aysal,barner}@mail.eecis.udel.edu

ABSTRACT

Quadratic Volterra filters are shown to be very effective in image sharpening applications. The linear combination of polynomial terms, however, yields poor performance in noisy environments. Weighted median filters, in contrast, are well–known for their outlier suppression and detail preservation properties. The weighted median sample selection methodology is naturally extended to the polynomial sample case, yielding a filter structure referred to as quadratic weighted median (QWM), that exploits the higher order statistics of the observed samples while simultaneously being robust to outliers arising in the higher order statistics of environment noise. The robustness of QWM filter to higher order statistics of noise is analyzed through the determination of breakdown probability. The simulation results show that the proposed method can successfully suppress the noise and enhance the image details simultaneously. Compared with the finite–impulse response (FIR) Quadratic Volterra sharpener, the QWM filter exhibits superior performance.

1. INTRODUCTION

Image sharpening is a classic problem in the field of image enhancement. A widely used simple approach for enhancing the blurred or imperfectly contrasted image is the unsharp masking. For instance, consider the structure in of Fig. 1. The input image is sent through a block that extracts edges and features. The output is then scaled by an appropriate factor k and added back to the original image. This method is generally referred to as unsharp masking [1] and is quite effective for enhancing low contrast images. The edge extraction block in Fig. 1 is often implemented as a linear highpass filter such as discrete linear Laplacian operator [1].

An apparent problem of this technique is that it does not discriminate between actual image information and noise. Thus noise is enhanced as well. To decrease the effects of this problem while still preserving the simplicity of the algorithm, the linear filter is extended to quadratic Volterra (QV) filter case [2].

QV filters can be described as a linear filter with higher order polynomial extensions. Even though the filter is not linear with respect to the input signal anymore, it is still linear in the impulse response coefficients, i.e., a linear combination of filters is equivalent to a filter with the same linear combination of the Kernel parameters. However, the polynomial nature of QV filter leads to poor performance in noisy environments. This poor performance results from the linear combination of polynomial terms utilized in such filters. Clearly, quadratic terms residing in the higher order kernels of the filter create outliers. The presented analysis shows that the tail heaviness of samples, and higher order terms contributing to a quadratic filter are well ordered, with the squared terms having heavier tails than cross terms and

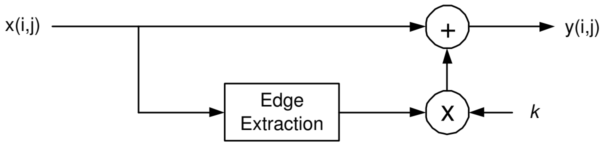

Figure 1: Block diagram of the unsharp masking technique.

cross terms having heavier than observation samples, i.e., $_{x_i} > _{x_ix_j} > _{x_i^2}$, where $_{x_i}$, $_{x_ix_j}$, and $_{x_i^2}$ denote the distribution tail decay rates of the observed samples (x_i), their cross–terms ($x_ix_j, i \neq j$), and squares (x_i^2), respectively. The heavier tails of the cross and squared terms indicate that robust methods for their sample combinations, rather than weighted sum, should be considered to avoid undue influence of outliers. In contrast to polynomial filters, weighted median (WM) filters are well known for their outlier suppression and detail preservation properties [3]. Indeed, WM filters are the optimal estimators of location, in a maximum likelihood (ML) sense, of samples characterized by the heavy tailed Laplacian distribution [3]. Hence, the WM sample selection methodology is naturally extended to the quadratic sample case, yielding the class of quadratic weighted median (QWM) filters, motivated by the presented linear, cross, and square term tail analysis for Gaussian statistics. The WM processing of cross and square term are also justified from a ML perspective [4]. The breakdown probability analysis demonstrates the improved robustness of the QWM filter class over traditional QV filters. The simulations carried out with applications to edge enhancement shows the superiority of the QWM structure over the FIR QV structure.

The remainder of this paper is organized as follows. In Section 2, the statistical foundations of QWM filter are presented. The traditional QV filter is introduced in Section 3, along with the derivation of the proposed QWM filter structure and its statistical analysis through the determination of the breakdown probability. Section 4 contains the simulations on image sharpening showing the superiority of the QWM filter over the FIR QV filter. Finally, the conclusions are drawn in Section 5.

2. STATISTICAL ANALYSIS

The effects of the product and square operators on a random variable's (RV) distribution's tail are considered noting that these statistics of cross-terms and square values are of particular interest in QV filtering. In this analysis we utilize the zero-mean Gaussian distribution,

$$_x(t) = \frac{1}{\sqrt{2}} \exp\left(-\frac{t^2}{2\ ^2}\right), \tag{1}$$

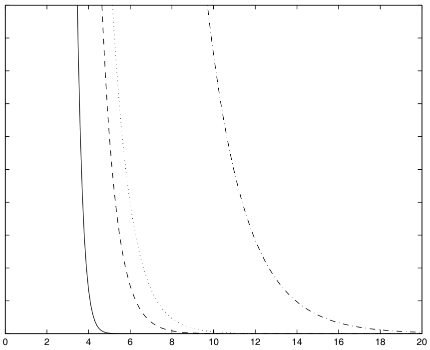

Figure 2: Tails of the $_X(\cdot)$ (solid), $_Z(\cdot)$ (dotted), and $_G(\cdot)$ (dash–dotted) density functions for the $_x = \;_y = 1$ Gaussian distribution case. Shown for reference is the $_X(\cdot)$ (dashed) Laplacian distribution with identical variance, $\;= 1/\sqrt{2}$.

where is the scale parameter. The theoretical probability density function (PDF) of a RV generated by squaring a Gaussian distributed RV X with scale parameter is given by

$$_G(t) = \frac{1}{\sqrt{2\;\; t}} \exp\left(-\frac{t}{2\;\;^2}\right), \qquad (2)$$

where $_G(t)$ denotes the PDF of the RV $G = X^2$. Also, the theoretical PDF of a RV generated by the product of two Gaussian distributed independent RV X and Y with scale parameters $_x$ and $_y$ respectively, is given by:

$$_Z(t) = \frac{K_0\left(\frac{|t|}{\;_x\;_y}\right)}{\;_x\;_y}, \qquad (3)$$

where $_Z(t)$ denotes the PDF of the RV $Z = XY$, and $K_n(\cdot)$ is the modified Bessel function of the second kind of order n. Note that, for large values of x, $K_0(x)$ behaves like $\frac{1}{\sqrt{x}}e^{-x}$ [5]. Then equation (3) can be approximated as,

$$_Z(t) \approx \frac{K}{\sqrt{|t|}} \exp\left(-|t|\right), \qquad (4)$$

for $_x = \;_y = 1$. Also, K is take as a constant that normalizes $_Z(t)$ to unity. The Gaussian PDF and (2) become $_X(t) = 1/\sqrt{2\;\;}\exp\left(-t^2/2\right)$ and $_G(t) = 1/\sqrt{2\;\;t}\exp\left(-t/2\right)$ respectively for $\;= 1$. For large $t > 0$ the arguments of the exponentials can be ordered as $-t^2/2 < -t < -t/2$, and the tail decay rate order is:

$$_X > \;_Z > \;_G, \qquad (5)$$

where $_X$, $_Z$, and $_G$ denote the tail decay rate of $_X(t)$, $_Z(t)$, and $_G(t)$.

The tails of $_X(\cdot)$, $_Z(\cdot)$, and $_G(\cdot)$ are shown in Fig. 2 for the Gaussian case with $_x = \;_y = 1$. Also shown for the reference is the tail of $_X(\cdot)$ for the Laplacian distribution case with identical variance obtained with $\;= 1/\sqrt{2}$, where is the scale parameter of the Laplacian distribution. As the figure shows, the tails exhibit the expected heaviness ordering, with the cross and square distributions having the heaviest tails. Also of note is that cross and square distributions tails are heavier than that of the median optimal Laplacian distribution. The heaviness of the tails indicates that the robust methods of sample combinations and output determination, rather than weighted sum, should be utilized to avoid undue influence of outliers and degradation of performance.

Having established the heaviness of the cross and square term distributions, we now consider the optimal combination of samples approached from a ML perspective considering Gaussian and heavy tailed Laplacian distribution.

Consider a set of N independent samples $x_1, x_2, \ldots, x_N$, each obeying a Gaussian distribution with a (possibly) different variances $_1^2, \;_2^2, \ldots, \;_N^2$. In this case, the ML estimate of the location parameter is determined by the minimization of

$$G_2(\;\;) = \sum_{i=1}^{N} \frac{1}{\;_i^2}(x_i - \;\;)^2, \qquad (6)$$

the solution to which is the weighted mean

$$\hat{\;} = \frac{\sum_{i=1}^{N} h_i x_i}{\sum_{i=1}^{N} h_i}, \qquad (7)$$

where $h_i = 1/\;_i^2 > 0$. This is simply a normalization of the standard FIR filter, $y = \sum_{i=1}^{N} h_i x_i$, where y is the output and the h_i terms are the FIR filter weights. Enforcing the positivity constraint on the weights constrains the resulting filters to be smoothers. In general practice, however, this constraint is relaxed, enabling FIR filters to take on a wide array of spectral characteristics.

A similar connection between filtering and ML estimation is established in the heavy tailed Laplacian distribution case [3]. The ML estimate of the location, in this case is, determined by minimizing

$$G_1(\;\;) = \sum_{i=1}^{N} \frac{1}{\;_i^2}|x_i - \;\;|. \qquad (8)$$

The solution to which is the weighted median:

$$\hat{\;} = MED(h_i \diamond x_i|_{i=1}^{N}), \qquad (9)$$

where $h_i = 1/\;_i^2 > 0$ and $\diamond$ is the replication operator defined as $h_i \diamond x_i = \overbrace{x_i, x_i, \ldots, x_i}^{h_i times}$. The weight positivity constrained again restricts the defined class of filters to smoothers, but, as in the FIR filter case, this constraint can be relaxed to enable more general filtering characteristics [3]. The filter output in the more general case is given by

$$y = MED(|h_i| \diamond sgn(h_i)x_i|_{i=1}^{N}), \qquad (10)$$

where $sgn(x) = 1$ when $x > 0$, $sgn(x) = 0$ when $x = 0$ and $sgn(x) = -1$ when $x < 0$.

Analogous relation between filtering and ML estimation is derived for cross and square terms in [4]. It has been justified from ML perspective that the WM processing of higher order statistics is more appropriate than weighted sum operators. These results, coupled with the preceding tail heaviness results, motivate the quadratic weighted median filters defined in the following section.

3. MEDIAN-TYPE QUADRATIC FILTERING

This section introduces the traditional QV filtering followed by the QWM filter derivation inspired by the statistical analysis presented in Section 2. The WM processing of cross and square terms are also justified from a ML perspective in [4].

A discrete–time QV filter is defined by [6]

$$y = C_2 \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} h_2(i_1,i_2) x_{i-i_1} x_{i-i_2} \,|_{i_1 \le i_2}, \tag{11}$$

where $h_2(i_1,i_2)$ is assumed to be a $N \times N$ upper triangular matrix (non–redundant terms) [7] representing the quadratic Volterra Kernel, and C_2 is a constant [7].

This formulation clearly indicates that, although the overall filtering operation is (polynomial) nonlinear, the filter output is linear with respect to the filter coefficients, and cross and square terms of observation samples. We are motivated to change the above weighted sum formulation, to a weighted median formulation by the results presented in Section 2. Recall that also in Section 2, it is shown that the linear combination of the samples is ML optimal only in the Gaussian distribution case, and that WM combinations are more appropriate in the heavier tailed case. Also, the heaviness of the distributions tails corresponding to cross and square terms indicate that robust combination methods should be utilized for higher order statistics. And lastly, under Gaussian distribution assumption, WM combinations for the cross and square terms are justified from a ML perspective in [4].

The quadratic weighted median (QWM) filter is therefore defined by replacing the weighted sum operators in (11) with weighted median operators,

$$y = C_2 MED(|h_2(i_1,i_2)| \diamond sgn(h_2(i_1,i_2)) x_{i-i_1} x_{i-i_2} |_{i_1=0}^{N-1}|_{i_2=0}^{N-1}). \tag{12}$$

Also, the QWM filter is expressed more compactly as

$$y = C_2 \langle \mathbf{h}_2 \rangle, \tag{13}$$

where we utilize the notation $\langle \mathbf{h} \rangle \equiv MED(|\mathbf{h}| \diamond sgn(\mathbf{h})\mathbf{x})$ and define $\mathbf{x} = [x_i^2, x_i x_{i-1}, \ldots, x_{i-1}^2, x_{i-1}x_{i-2}, \ldots, \ldots, x_{i-N+1}^2]$.

A direct measure of filter robustness is given by the breakdown probability, which is defined as the probability of an impulse occurring at the filter output [8]. The breakdown probability of selection type filters, such as WM filters, can be established utilizing the sample selection probability (SSP), defined as the probability that the filter output is the i-th sample, i.e., $s_i = P(y = x_i)$, $i = 1, 2, \ldots, N$. The SSPs can be established for any WM filter with integer valued weights [8] and, as noted above, any WM filter with real valued weights can be represented by an equivalent WM filter with integer valued weights [8]. Thus the SSPs can be established for any WM filter, and we set $\mathbf{s} = [s_{1,1}, s_{1,2}, \ldots, s_{2,2}, s_{2,3}, \ldots, s_{N,N}]$, as the SSP vector for the QWM filter. Let p and $\mathcal{N}(p) = P(y \neq \pm\ )$ be the probability that an observed sample is corrupted by an impulse, i.e., $p = P(x_i = \pm\ )$ and the probability that there is no breakdown, respectively. Then the BDP of a QWM filter is given by

$$\mathcal{N}(p) = \sum_{i=1}^{N} \sum_{j=1}^{N} s_{i,j}(1-p)^2 \,|_{i<j} + \sum_{i=1}^{N} s_{i,i}(1-p), \tag{14}$$

where N is the filter order. Thus, the BDP of QWM is simply $(p) = 1 - \mathcal{N}(p)$.

The weighted sum methodology of traditional QV filtering produces an impulsive output whenever one or more impulses are present in the *observations* set, yielding a breakdown probability of $_V(p) = 1 - (1-p)^N$ that is equivalent to a standard window size N linear FIR filter. That is, the QV filter breakdown probability is strictly a function of window size and is independent of polynomial order and coefficient values. The breakdown probabilities for QWM and QV filters with $N = 5, 7$, and 9 are plotted in Fig. 3. Samples are uniformly weighted in the QWM filter case, which yields the most robust performance. As the figure shows, the QWM filter has a lower breakdown probability than the QV filter in all cases. Moreover, increasing the window size significantly increases the breakdown probability of the QV filter, while the QWM filter breakdown probability is largely invariant to changes in window size.

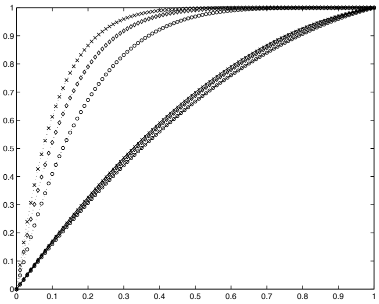

Figure 3: Breakdown probability for the QWM (solid) and QV (dashed) filters with window sizes $N = 5\,\text{(circle)}, 7\,\text{(diamond)}$, and 9 (cross).

4. SIMULATION RESULTS

In this section, the performance of QWM filter is compared with the QV filter given in [2] as

$$\begin{aligned} y(i,j) = {} & 3x^2(i,j) - 0.5x(i+1,j+1)x(i-1,j-1) \\ & - 0.5x(i+1,j-1)x(i-1,j+1) - x(i+1,j)x(i-1,j) \\ & - x(i,j+1)x(i,j-1), \end{aligned} \tag{15}$$

for image sharpening application. This filter is used in the edge extraction block given in Fig. 1 and is shown to perform better than the linear unsharp masking systems [2]. Hence, we will compare the performance of the QWM filter with QV filter defined in (15). It has been shown that [3] a WM filter with the weights same as a FIR filter possess similar spectral responses. Hence in the QWM filter formulation, the filter weights in (15) are utilized. Thus, replacing the weighted sum operations in (15) with WM operators yields the QWM counter–part,

$$\begin{aligned} y(i,j) = {} & MED(3 \diamond x^2(i,j), -0.5 \diamond x(i+1,j+1)x(i-1,j-1), \\ & -0.5 \diamond x(i+1,j-1)x(i-1,j+1), -1 \diamond x(i+1,j)x(i-1,j) \\ & -1 \diamond x(i,j+1)x(i,j-1)). \end{aligned} \tag{16}$$

The "Lena" image of size 512×512 shown in Fig. 4 (a) (zoomed in) is used in our experiment. The filter performances are both tested in noise–free and Gaussian noise environments. The scaling factors in QV and QWM filter cases are chosen so that the same enhancement level is achieved. Fig. 4 compares image sharpening results utilizing QV and QWM based unsharp masking operating on

(a)

(b)

(c)

(d)

(e)

(f)

Figure 4: (a) The original "Lena" image, (d) noisy "Lena" corrupted by Gaussian noise, (b), and (e) sharpened "Lena" images using QV filter, (c), and (f) sharpened "Lena" images using QWM filter.

noisefree and additive Gaussian noise contaminated images. Inspection of the images shows that the QWM output contains sharper edges, with less overshoot and ringing, as well as more consistent uniform areas. These effects are especially noticeable in the Gaussian noise corrupted case. Even though the image in this case is corrupted by a low level of noise, the noise affects are amplified in the traditional QV system. The QWM results, in contrast, minimizes the influence of cross and square term outliers, resulting in crisp edges, more consistent uniform regions, and visually more pleasing results.

5. CONCLUSIONS

In this paper, the statistics of the higher order terms contributing to the output of a quadratic Volterra (QV) filter is analyzed. The results indicate that even though the observation samples are Gaussian distributed, the higher order terms exhibit heavy tail distributions, even heavier than median optimal Laplacian distribution. A novel quadratic weighted median (QWM) filter structure is proposed. The robustness of QWM filter to higher order statistics of noise is analyzed through the determination of breakdown probability. Simulations carried out to evaluate and compare the performance of the proposed structure (QWM) shows the superiority of QWM over the traditional QV filtering.

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