MULTI-CARRIER SIGNAL SHAPING EMPLOYING HERMITE FUNCTIONS

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ABSTRACT

In this paper, we introduce a novel signal shaping approach for multi-carrier systems. We propose to combine Hermite functions in order to get a good time-frequency localization property for multi-carrier signals, which is important for robustness against the time-frequency dispersion of the wireless channel. In our work, we orthogonalize the linear combination of Hermite functions in order to get a Weyl - Heisenberg set for multi-carrier signal shaping. We disprove the conjecture that optimum signal shape is achieved by orthogonalizing the Gaussian signal, which results in the IOTA signal. We show that the time-frequency localization of the proposed signal is better than that of the IOTA signal.

1. INTRODUCTION

Multi-carrier (MC) transmission, in particular orthogonal frequency division multiplexing (OFDM), is employed recently in various systems mainly due to its robustness against channel distortions [1]. MC systems are lattice structures formed by time-frequency shifts of a prototype function such as a rectangular pulse or a raised cosine pulse. These systems divide the frequency selective wideband channel into small flat fading subchannels [2]. Hence very large bandwidths can be employed easily due to reduced effect of channel dispersions.

There are two types of channel dispersion in wireless systems: frequency dispersion and time dispersion. The first type is due to Doppler spread and the latter is due to multi-path propagation. In OFDM systems, it is straightforward to combat time dispersion by adding a cyclic prefix to the symbols. Although cyclic prefix provides the protection against multi-path delay spread or channel dispersion in time, OFDM systems are vulnerable to dispersion in frequency, namely the Doppler spread. Since the subbands of the OFDM systems are very densely packed, with the increased speed of receiver and transmitter with respect to each other, orthogonality between sub-carriers are disrupted and significant performance degradation is faced. This is one of the major challenges of OFDM systems that are meant for the nomadic applications. This is also the reason behind the application of OFDM mainly to the fixed wireless systems. The mobility issue is also investigated under the frame work of the nomadic wireless broadband standard work groups; IEEE 802.16e [3] and IEEE 802.20 [4]. Both IEEE 802.16e and IEEE 802.20 employ OFDM architecture, hence the increase in the mobility performance of OFDM systems is critical for the future development of 4G networks.

In order to get a good localization property, the signal shapes employed in the carriers should decay very fast both in time and frequency. The localization problem in the OFDM systems is due to slow decaying property of the raised cosine pulses or the square pulse that are employed. Hence other signal shapes are considered in the literature [5, 6].

The time-frequency shifts of the signals should be orthogonal for the detection process of the OFDM signals. Although the Gaussian pulse has the best localization property in both time and frequency, it does not match the orthogonality criteria required by the OFDM systems. Hence transforming the Gaussian pulse into an orthogonal pulse with a slightly worse localization was proposed in [5] as the so called IOTA approach. IOTA has very close localization to the Gaussian signal, yet it is orthogonalized in time and frequency.

Another approach is to orthogonalize the Gaussian signal by combining it with other Hermitian signals . In [6], five Hermitian signals including the Gaussian signal were combined in order to obtain an orthogonal signal. However the results were limited to symmetric Hermitian signals and orthogonality is not guaranteed for any lattice structure.

From Gabor theory it is well known that for unit density structures, a well localized orthogonal set cannot be achieved due to Balian-Low theorem [7, 8]. Hence in the literature many structures employing half lattice density, so called OFDM Offset-QAM (OFDM/OQAM) approaches were proposed [9, 10].

All of the above mentioned approaches assume that the best localized orthogonal signal can be achieved by orthogonalizing the best localized nonorthogonal signal, namely the Gaussian signal. In this work, we prove that this is incorrect. It is shown that by combining Hermite functions and orthogonalizing them, an orthogonal signal which is better localized than the IOTA signal can be achieved. The proposed method is a general framework, since the first Hermite function is the Gaussian signal, and the set of Hermite functions is an orthogonal set that includes the most localized signals which are orthogonal to the Gaussian signal [11].

The organization of this paper is as follows: In Section 2, we present the MC system that employs signal shaping and give the necessary conditions for orthogonality. In Section 3, we compare the orthogonalized Gaussian and Hermite functions. Simulation results are presented in Section 4 and conclusions are given in Section 5.

2. ORTHOGONAL MULTI-CARRIER SYSTEMS

In order to represent a multi-carrier signal space in frequency and time, we consider the rectangular lattice structure given in Fig. 1. The carriers are equally spaced in time and frequency. If the frequency spacing is given as F and time spac-

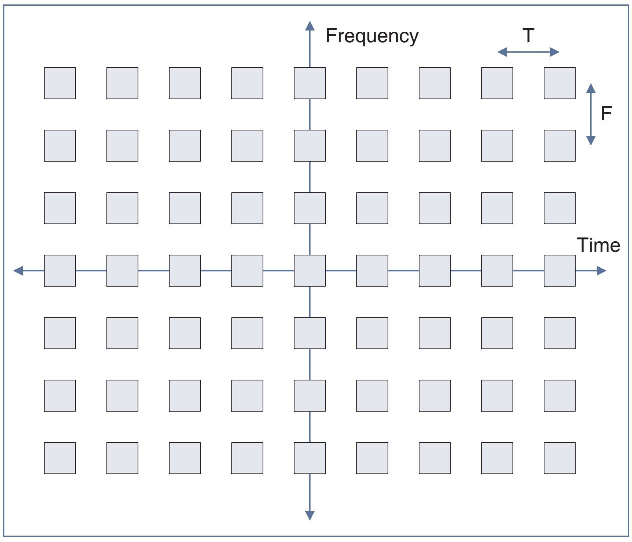

Figure 1: Distribution of orthonormal carriers in time-frequency lattice for rectangular lattice structures.

ing is given by T, then we can define the density of the lattice as $\Delta = 1/FT$. A carrier signal $\varphi(t)$ is employed such that its time-frequency shifts;

$$\varphi_{mn}(t) = \varphi(t - nT)\exp(j2\pi mFt), \tag{1}$$

form an orthonormal set, where m and n are integers. These orthonormal set of functions form an OFDM system. The functions $\varphi_{mn}(t)$ are also called the coherent states since they are generated by time-frequency shifts from a single function $\varphi(t)$ [12].

For such a set of functions $\varphi(t)$, the orthonormality condition can be written as:

$$\int_{-\infty}^{\infty} \varphi_{kl}^{*}(t)\varphi_{mn}(t)dt = \delta_{km}\delta_{ln}. \tag{2}$$

Due to lattice translation invariance, with respect to the central signal φ_{00} we can rewrite (2) as:

$$\int_{-\infty}^{\infty} \varphi_{00}^{*}(t)\varphi_{mn}(t)dt = \delta_{m}\delta_{n}. \tag{3}$$

The set of functions φ_{kl} is referred to as a Weyl-Heisenberg set [11]. Although many orthogonal sets satisfying (3) can be built, the performance of the MC system should also be robust under time or frequency dispersion. For this reason the functions $\varphi(t)$ must be well localized in time and frequency.

It is not trivial to find such a well localized orthogonal function intuitively, so the orthogonalization of well localized nonorthogonal signals were proposed in [5, 9]. However, in all of the cases that are presented in the literature, the well localized nonorthogonal functions, which we will refer as the mother function from this point on, is chosen as the Gaussian function. The orthogonalization of Gaussian function results in the IOTA function of [5]. In the following section, we prove that this is not the optimal case.

3. COMPARISON OF MOTHER FUNCTIONS

The Gaussian function is the most localized function in terms both time localization (T_l) and frequency localization (W_l) and the localization is given by $T_l W_l$, where the localizations are defined by

$$\begin{aligned} T_l &= \sqrt{\int_{-\infty}^{\infty} t^2 |\psi(t)|^2 dt}, \\ W_l &= \sqrt{\int_{-\infty}^{\infty} f^2 |\Psi(f)|^2 df}. \end{aligned} \tag{4}$$

where $\Psi(f)$ is the Fourier transform of $\psi(t)$. So for any signal $\psi(t)$, uncertainty principle is satisfied as:

$$T_l W_l \geq \frac{1}{4\pi}. \tag{5}$$

The main motivation of employing the Gaussian signal is that when the Gaussian mother function is orthonormalized, the resulting orthonormal function $\psi(t)$ will also have a localization very close to the Gaussian one. If we employ the Gaussian signal:

$$\psi(t) = e^{-\pi t^2} \tag{6}$$

and orthonormalize it for the half density lattice, $F = T = \sqrt{2}$, the resulting orthonormalized signal $\phi(t)$ is the IOTA function.

As an alternative approach, we propose to combine Hermite functions for building the mother function. Then we orthogonalize this novel mother function by any method proposed in the literature. We form the mother function $\phi(t)$ as:

$$\phi(t) = \sum_{k=0}^{\infty} a_k H_k(t), \tag{7}$$

where H_k is the k^{th} Hermite function. The main motivation behind employing Hermite functions is that the first Hermite function is the Gaussian function given by (6) and k^{th} Hermite function is the k^{th} best localized function that is orthogonal to the Gaussian and $k-1$ other Hermite functions.

In this work, we tried to optimize these coefficients by brute force approach, using exhaustive search in order to maximize the localization. There are three main results that are obtained from the simulations:

- The coefficients a_k decrease exponentially with increasing k, hence only 5 coefficients were satisfactory for convergence.
- Only the Hermite functions whose frequency transforms are the same as themselves, hence time-frequency symmetric Hermite functions give good localization performance. These Hermite functions are the ones where $k = 0(mod\ 4)$.
- When we set $a_0 = 1$, all other coefficients a_k should be real. In general all complex coefficients must have the same phase for optimum localization with the exception of a 180 degree phase shift.

Keeping these conditions in mind, the Hermite functions

Table 1: Optimum coefficients for Hermite functions

a_0	1
a_4	-1.5×10^{-3}
a_8	-3×10^{-6}
a_{12}	-2×10^{-10}
a_{16}	-2×10^{-13}

that are employed are given as:

$$\begin{aligned}
H_0(t) &= G(t)\\
H_4(t) &= 16G(t)(4\pi^2t^4-6\pi t^2+.75)\\
H_8(t) &= 256G(t)(16\pi^4t^8-112\pi^3t^6+210\pi^2t^4\\
&\quad -105\pi t^2+105/19)\\
H_{12}(t) &= 4096G(t)(64\pi^6t^{12}-1056\pi^5t^{10}+5940\pi^4t^8\\
&\quad -13860\pi^3t^6+\frac{51975}{4}\pi^2t^4-\frac{31185}{8}\pi t^2+\frac{10395}{64})\\
H_{16}(t) &= 65536G(t)(256\pi^8t^{16}-7680\pi^7t^{14}+87360\pi^6t^{12}\\
&\quad -480480\pi^5t^{10}+1351350\pi^4t^8-1891890\pi^3t^6\\
&\quad +1182431.25\pi^2t^4-253378.125\pi t^2+\frac{2027025}{256}).
\end{aligned}$$

where $G(t)=\exp(-\pi t^2)$. The optimum coefficients of these functions are given in Table 1.

If we form our mother function by the coefficients given in Table 1 and then orthogonalize it, then the resulting orthogonal signal is given in Fig. 2. In this figure, the Gaussian signal and the orthogonalized Gaussian signal (IOTA) are included as well. Although the functions are symmetric in time

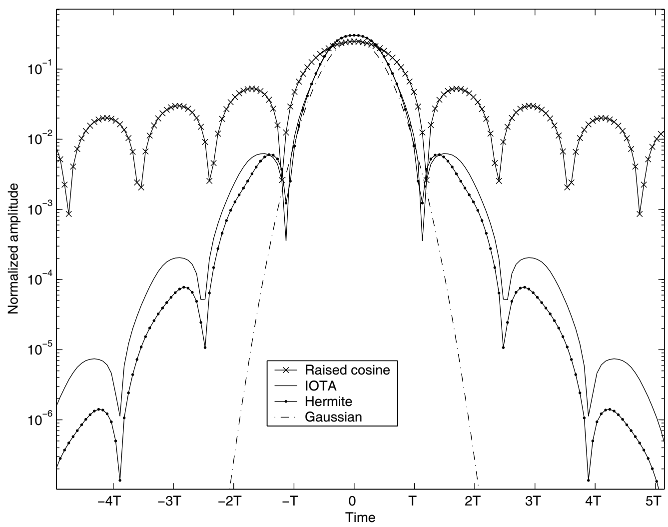

Figure 2: Comparison of the proposed signal with square pulse, raised cosine with roll-off 0.1, IOTA and the Gaussian signal

and frequency, we include the frequency response of Fig. 3 for an extended range, in order to get a broader comparison between IOTA and our signal.

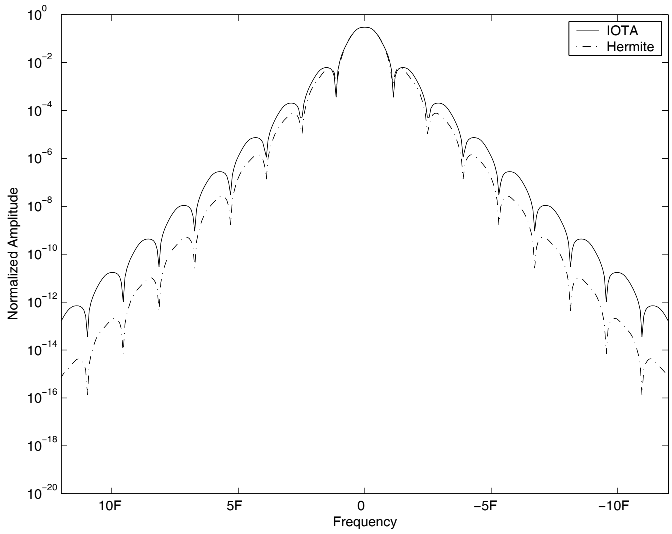

Figure 3: The proposed orthonormalized waveform vs. the IOTA signal in frequency domain.

As it is clearly seen in Fig. 2 and Fig. 3, the proposed orthogonal signal is much more localized in both time and frequency when compared to the raised cosine pulse and even to the IOTA pulse which has been assumed to be the optimum one. Square pulse is not included in the figure, since it has the worst localization when compared to the other pulse shapes. Numerically, the time-frequency localization of a Gaussian pulse is $1/4\pi = 0.0796$, the localization of IOTA is 0.0815 and the proposed method has a localization of 0.0811. Also in Fig. 2 and 3, it can be seen that the proposed signal has wider nulls and smaller sidelobes when compared to IOTA, which makes it more robust to the time and frequency dispersion effects of the channel.

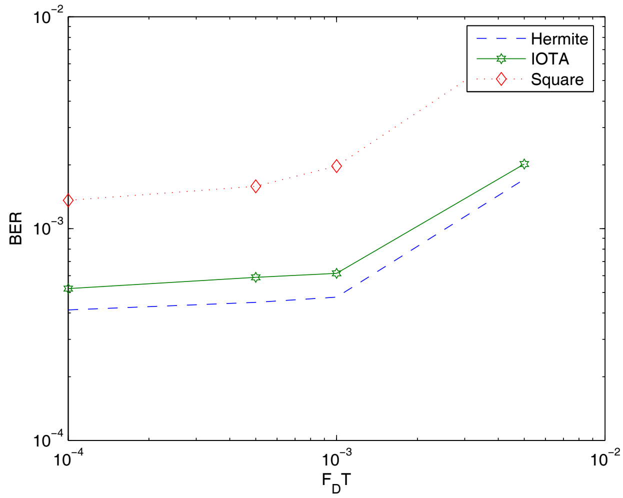

Figure 4: BER vs. normalized Doppler spread curves for 32 subcarriers for SNR of 12 dB.

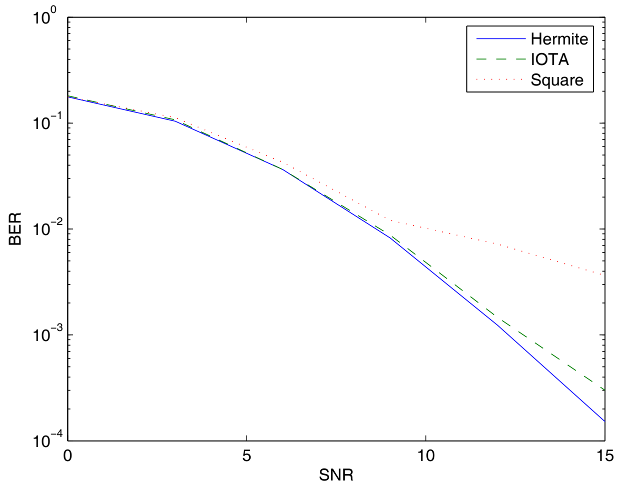

Figure 5: BER vs. SNR curves for 32 subcarriers for $f_D T = 0.0005$.

4. THE EFFECT OF INTER-CARRIER INTERFERENCE

Next, the effect of Doppler spread is investigated. One of the weakest points of the OFDM systems is that they are not robust in the high mobility. The reason behind this draw-back is that for high speed users with a high Doppler spread, the orthogonality between the subcarriers is defected, resulting in inter-channel interference (ICI). However, by employing a pulse shape which rapidly decays in frequency, the amount of ICI can be reduced.

In our simulation model, we use a Rayleigh fading channel with an SNR of 12 dB. We concentrate on ICI and do not consider the effect of multi-path in these simulations. We measure the mobility as the product of Doppler frequency (f_D) and symbol time (T) as $f_D T$. Fig. 4 shows the effect of speed on the performance of OFDM systems with different pulse shapes. As it is depicted in the figure, Hermite and IOTA based OFDM systems are much more robust to speed, the former being more robust than the latter.

Next, we look at the effect of the SNR. In the next simulation we assume that the product $f_D T = 0.0005$. As shown in Fig. 5, the performance gain by application of Hermite functions is visible and solid.

All of these results show that the robustness of the OFDM systems to the Doppler spread is increased when Hermite pulse shapes are introduced. The overall performance of the Hermite pulse is much higher than the square pulse and marginally better than the IOTA pulse.

5. CONCLUSIONS

In this work, we employed Hermite functions for the construction of time and frequency dispersion resistant MC signals. It is shown that the conjecture of IOTA function as the optimum localized signal shape for MC systems is incorrect. It is possible to find a better localized orthogonal signal set by employing Hermite functions.

We have proved that by orthogonalizing a linear combination of Hermite functions, very well localized orthogonal pulse shapes can be achieved. We have also shown that these pulse shapes achieve better localization results than IOTA.

As a result of the better localization property, the MC system that employs the proposed signal shape becomes more robust to multi-path fading and Doppler spread introduced by wireless channels.

It is verified by simulations that the proposed pulse shapes provides better BER performance for wireless mobile channels when compared to other pulse shapes. It is also shown that this superiority prevails as the mobile speed and Doppler spread increases. This proves to be a very critical improvement for broadband wireless access systems which are limited by user speed for high data rate data transmission.

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