CUBIC PHASE COUPLING ESTIMATION VIA TRISPECTRUM AUTORREGRESIVE MODELING

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ABSTRACT

In this communication the use of AR modelling in the estimation of cubic phase coupling is studied. After obtaining the trispectrum of an harmonic signal with cubic phase couplings, the parametric modelling is used to design a method that models this trispectrum and allows the locatation of the cubic couplings. This method is based on a twofold AR modelling of the data using their fourth-order cumulants, such as, after the extraction of the two sets of coefficients, called frequency and sum coefficients, the modelled trispectrum is built and the phase coupling are located as the peaks in this trispectrum. A frequency selection method is proposed to overcome the problems asociated to the maximum-detection algorithms in the trispectral domain, which also reduces significantly the computational burden necessary to find these maxima. Practical conditions of implementation and the limitations of using this approach are showed and discussed in simulations.

1. INTRODUCTION

In some sitiations, signals with harmonic componentes present contributions at frequencies that are sums and/or differences of other components. When such type of phenomenon are the result of n-order nonlinearities, this is called phase coupling. Specifically, it is said that $n+1$ terms of a harmonic signal constitute a phase coupling when one of its frequencies is the sum of the others and its phase is, at the same time, the sum of the others phases [3]. If only the first of the two previouse conditions occurs then it is said that the $n+1$ frequencies are harmonically related. In certain applications [3] it is necessary to find out if $n+1$ harmonically related frequencies are actually phase coupled. An example of this is the separation between direct and multiple scattering when two or more echo centers interacting themselves in radar scattering [5]. Due to the fact that the power spectrum removes all phase information, a second order analysis of this signal cannot solve the problem and it is necessary to use n-order statistics. Although quadratic phase coupling has received great interest in the literature and bispectrum autorregresive modelling has been used in practical problems to obtain the coupled frequencies [4], the cubic case has received less attention, principally due to the computational complexity that a direct estimation of the trispectrum (the spectral function that characterize the fourth-order statistic) possesses. This communication analyses the detection of cubic phase coupling in harmonic signals using autorregresive (AR) modelling of the trispectrum, which reduce greately the computational complexity) and proposes a practical procedure to obtain the coupled frequencies.

In this communication, after the introductory section, the signals are defined in Section II, where their fourth-order cumulants and trispectrum are also calculated. The AR modelling is used in Section III to build a trispectrum with the same maxima that the real one. An efficient frequency detection method that avoids the search of maxima in the tridimensional domain of the trispectrum is presented in this section as well. Two different AR methods are studied in Section IV via simulations, where the ability of the proposed approach to detect and locate the cubic phase couplings are tested. The communication finishes with the principal conclusions.

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2. CUBIC PHASE COUPLING

Let us consider a harmonic signal such as:

$$x(n) = \sum_i A_i e^{j(\nu_i n + \psi_i)} \tag{1}$$

where the amplitudes A_i are positive real constants, the frequencies ν_i are constant in the interval $[0, 2\pi)$ and the phases ψ_i are random variables uniformly distributed in $[0, 2\pi)$. It is said that four frequencies form a cubic phase coupling when:

$$\nu_4 = \nu_1 + \nu_2 + \nu_3 \text{ and } \psi_4 = \psi_1 + \psi_2 + \psi_3 \tag{2}$$

So that, distinguishing the contribution of cubic couplings, the above signal can be written as:

$$x(n) = \sum_{k=1}^{t} \sum_{i=1}^{4} A_{ki}^c e^{j(\nu_{ki}^c n + \psi_{ki}^c)} + \sum_{i=1}^{r} A_i^u e^{j(\nu_i^u n + \psi_i^u)} \tag{3}$$

with:

$$\nu_{k4}^c = \nu_{k1}^c + \nu_{k2}^c + \nu_{k3}^c \text{ and } \psi_{k4}^c = \psi_{k1}^c + \psi_{k2}^c + \psi_{k3}^c \tag{4}$$

for $k = 1, ..., t$.

The $4t$ frequencies in the first term correspond to t cubic phase couplings, while the r frequencies of the second term constitute others possible frequencies, which either are not coupled or are not cubically coupled (but can be quadratically, for example).

Power spectrum of this signal treats both kinds of frequencies in the same way, therefore it cannot distinguish when four frequencies form a cubic phase coupling or when they only are harmonically related.

However, the fourth-order statistic of this signal is indeed able to distinguish both kinds of frequencies. This can be shown through the fourth-order cumulant series of this signal:

$$\begin{aligned} c_4^x(\tau_1, \tau_2, \tau_3) &= Cum[x^*(n), x(n+\tau_1), x(n+\tau_2), x(n+\tau_3)] \\ &= \sum_{k=1}^{t} A_{k1}^c A_{k2}^c A_{k3}^c A_{k4}^c \sum_{(j_1, j_2, j_3) \in S_3} e^{j(\nu_{k1}^c \tau_{j_1} + \nu_{k2}^c \tau_{j_2} + \nu_{k3}^c \tau_{j_3})} \end{aligned} \tag{5}$$

where S_3 is the set of all possible permutations of the set $(1, 2, 3)$ and $x^*(n)$ represent the complex conjugate of $x(n)$. Likewise, the trispectrum of this signal is easy calculated and the result is:

$$\begin{aligned} T_4^x(\omega_1, \omega_2, \omega_3) &= \sum_{k=1}^{t} A_{k1}^c A_{k2}^c A_{k3}^c A_{k4}^c \\ &\cdot \sum_{(j_1, j_2, j_3) \in S_3} \delta(\omega_1 - \nu_{kj_1}^c) \delta(\omega_2 - \nu_{kj_2}^c) \delta(\omega_3 - \nu_{kj_3}^c) \end{aligned} \tag{6}$$

From this equation, it can be seen that the trispectrum of a signal following (3) is null in all its domain except in those frequencies corresponding to the individual frequencies of the cubic phase couplings, where it will be infinite. Therefore, a peak in the trispectrum indicates the existence of a cubic coupling, and its coordinates indicate the individual frequencies of this coupling.

As the direct estimation of the trispectrum using Fourier type methods needs a big calculus esfort and possesses great limitations in the resolution [3], AR modelling is used to build a trispectrum with similar properties than the real one that appears in (6). This modelling looks for an adequate reproduction of the peaks of the trispectrum, so it is not necessary that modeled and the real trispectrum be equal out of the peaks. This happens in a lot of AR modelling applications, like in speech, where is the goal is to model correctly the peaks but not the valleys in the power spectrum. The way to achieve it in an efficient manner is showed in the next section.

3. AR MODELLING OF CUBIC PHASE COUPLING

In order to apply AR modelling to the signals given in (3), two sets of AR coefficients are defined in this section. With these coefficients, that verify certain linear systems of equations, a trispectrum that models properly the one in (6) is built. The first of these systems is:

$$\sum_{i=0}^{p} a_1(i) c_4^x(m-i,n,q) = 0 \tag{7}$$

for $n,q = 0,\dots,p$ and $m = 1,\dots,p$, where $p \geq 3t$, and t being the total number of coupled frequencies. In order this first set of coefficients, called set of frequency coefficients, to hold the following expression must be satisfied:

$$A_1(\omega)|_{\omega=\nu_{ks}^c} = \sum_{i=0}^{p} a_1(i) e^{-j\nu_{ks}^c i} = 0 \tag{8}$$

for $k = 1,\dots,t$ and $s = 1,2,3$. This implies that the Fourier transform annuls at the individual coupled frequencies. Therefore, the transfer function yields an infinite contribution at these frequencies.

The second set of coefficientes, called set of sum coefficients, is the one that satisfies the system of equations:

$$\sum_{i=0}^{p} a_2(i) c_4^x(m-i,n+m-i,q+m-i) = 0 \tag{9}$$

for $n,q = 0,\dots,p$ and $m = 1,\dots,p$, where $p \geq t$ and t again being the number of couplings. As it happened for the frequency coefficients, (9) will hold true if and only if the following equation is satisfied:

$$A_2(\omega)|_{\omega=\nu_{k4}^c} = \sum_{i=0}^{p} a_2(i) e^{-j\nu_{k4}^c i} = 0 \tag{10}$$

for $k = 1,...,t$. The above means that the Fourier transform of the sum coefficients has zeroes precisely at the sum coupled frequencies, therefore its transfer function will yield an infinite contribution at these frequencies.

Once the systems of equations are solved, the sets of coefficients are obtained. Using the transfer functions of these sets of coefficients, $H_1(\omega) = 1/A_1(\omega)$ and $H_2(\omega) = 1/A_2(\omega)$, the trispectrum can be modeled using the expression:

$$T(\omega_1,\omega_2,\omega_3) = H_1(\omega_1)H_1(\omega_2)H_1(\omega_3)H_2^*(\omega_1+\omega_2+\omega_3) \tag{11}$$

The four factors of the above trispectrum yield an infinite contribution when the frequencias are equal to the individual frequencies of a cubic phase coupling; i.e.,the above holds when:

$$(\omega_1,\omega_2,\omega_3) = (\nu_{k1}^c, \nu_{k2}^c, \nu_{k3}^c) \tag{12}$$

for $k = 1,\dots,t$. In practice, equations (8) and (10) will not be exact, but if the estimation is accurate enough, then it is expected that Fourier the transforms of the sets of coefficients evaluated at the corresponding frequencies will approach zero. This will make the defined spectrum (11) when evaluated at single coupling frequencies much higher than in any other point of its domain, which implies the peaks presence at frequencies corresponding to a cubic phase coupling, as happened in theoretical trispectrum (6). Although (6) and (11) are not exactly identical, both show the same peaks distribution, which makes peaks detection based on AR modelling possible.

Specifically, in order to locate the couplings with AR modelling, first the fourth-order cumulants of the signal $x(n)$, given in (3), should be estimated. This input signal can be corrupted with additive gaussian noise of unknown spectrum, which does not affect theoretically the value of the cumulants, since the cummulant of a sum of signals is the sum of the cumulants and the cumulant of fourth-order of a gaussian process is zero [3]. With these cumulants, the systems of equations (7) and (9) are formed and solved, obtaining both sets of AR coefficients, frequency ones and sum ones. Using both sets of coefficients, the trispectrum given in (11) can be built. The couplings will be the peaks of this formed trispectrum. The peaks coordinates match the single frequencies of cubic phase coupling. However, this procedure requires a three variable function peaks finding, which can be computationally intensive.

For example, if the sampling frequency is set to unity, then Fourier transform ranges from 0 Hz to 1 Hz. If 0.01 Hz frequency precision is required at estimations, a minimum of 100 points are necessary in the Fourier transforms (8) and (10) to form (11). This implies a 10^6 points trispectrum, in which the peaks of a three dimensional function should be found. If more precision is needed, the number of points in the transforms neccesary to compute the trispectrum makes the calculation time prohibitive. Beside this, spectral estimation errors, like leakage, can only be overcome by increasing the total amount of data points for the Fourier transforms.

The above problem makes the trispectrum peaks finding in (11) a quite hard computing task. However, another scheme can be used to detect coupled frequencies by using the transforms of both set of coefficients, (8) and (10). Once the set of coefficients are obtained solving (7) and (9), the zeros of the Z transform of the coefficients can be calculated:

$$A_1(z) = \sum_{i=0}^{p} a_1(i) z^{-i} \text{ and } A_2(z) = \sum_{i=0}^{q} a_2(i) z^{-i} \tag{13}$$

If $p = 3t$ and $q = t$, zeros in $A_1(z)$ appear at $z = e^{j\nu_{ks}^c}$ for $k = 1,...,t$ and $s = 1,2,3$, i.e., at individual coupled frequencies; while the zeroes of $A_2(z)$ are at $z = e^{j\nu_{k4}^c}$ for $k = 1,...,t$, i.e., sum coupled frequencies. With the first set of zeroes, individual frequencies are obtained, but we have no knowledge of how group these invidual frequencies in groups of three to form each one of the couplings. The second set of zeroes is used to do this, since the sume of the individual frequencies in each couplings should be equal to the sum frequency. By this scheme, trispectrum peaks finding is avoided and there is not precision counterbacks, due to the fact that frequencies are obtained by calculating the roots of a polynomial expression.

A standard approach to improve the harmonic component determination is to overdetermine the problem to solve, which means increasing the index p and q in (13). Doing this, the calculated zeros correspondent with the frequencies of the couplings are much closer to their theoretical values, but new spurious zeros are found which do not correspond to any coupled frequency. So, when $p \geq 3t$, zeros corresponding to individual coupled frequencies are to be identified between all the set of zeros (that are more than the individual coupled frequencies due to the overdetermination). This can be accomplished identifing the $3t$ zeros as those closer to the unit circle. However, as there is no constrains in the location of spurious zeros, so they can be found all over the complex plane, there is a certain probability of a spurious zero to be closer to the unit circle than a correct one, producing a mistaken identification. The chance for this to happen depends on the number of data points, the method

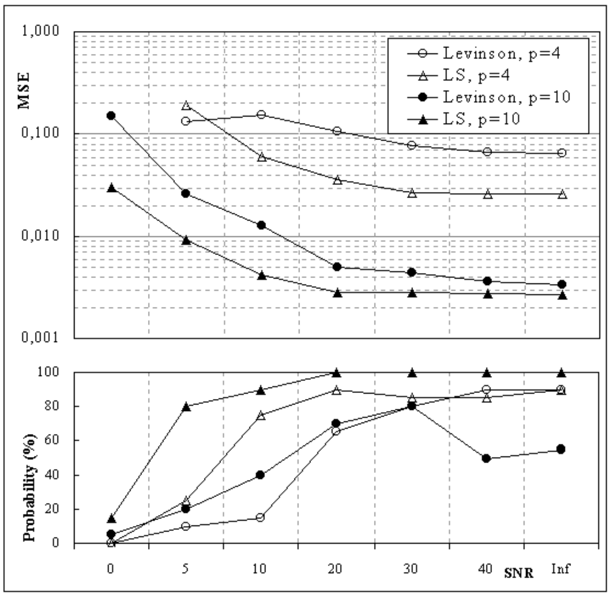

Figure 1: MSE and probability of correct zeros classification for Levinson and LS method and for $p = 4$ and $p = 10$.

used for solving the system of equations, the order p, the signal corrupting noise level and the signal itself. The effect of all these factors will be study in the next section in simulations.

4. RESULTS

In order to study the behaviour of the proposed AR based scheme in the detection of cubic phase coupling, two different examples are going to be analysed. In the first one, only a cubic phase coupling appears, while in the second example, together with one phase coupling, there are four frequencies harmonically related but not phase coupled.

Example 1. The signal is given by:

$$x(n) = \sum_{i=1}^{4} A_i^c e^{j(\nu_i^c n + \psi_i^c)} \tag{14}$$

where the frequencies verifies that $\nu_4^c = \nu_1^c + \nu_2^c + \nu_3^c$, and phases $\{\psi_k^c,\ k = 1,2,3\}$ are i.i.d. random variables uniform in the interval $[0, 2\pi)$, with $\psi_4^c = \psi_1^c + \psi_2^c + \psi_3^c$ and the frequencies take the values:

$$(\nu_1^c, \nu_2^c, \nu_3^c, \nu_4^c) = 2\pi(0.07, 0.12, 0.45, 0.64) \text{ (rad/s)} \tag{15}$$

In the estimation procedure, 64 independent records with 64 data values each has been used. Additive colored gaussian noise with a specific SNR is added to the signal. Whis this signals, the fourth-order cumulants are estimated, computing the cumulants in each record and then taking the final value for the cumulantes the mean of all the cumulants in each record. With these cumulants, the system of equations (7) is built and then is solved using the Levinson method [2] and the standard Least Square (LS) procedure. The first method builds a determined Toeplitz system of equations setting $n = q = 0$, and then the coefficients are obtained using the iterative Levinson procedure for Toeplitz systems of equations. The second method builds an overdetermined system setting $q = 0$ and $n = 1, \ldots, p$, then the least-squares procedure is applied to obtain the frequency coefficients.

Two indexes or parameters are calculated in the simulations to characterize the behaviour, which are shown in Figure 1. The first parameter is the probability of correct classification of the zeroes, shown in the bottom graph. This parameter is computed counting the times there is not spurious zeros between the $3t$ selected frequencies in 20 independent realizations, i.e. the times, in 20 realizations, that the zeros correspondent with the individual frequencies are closer to the unit circle than any spurious zero are counted, and with this number the correct classification probability is estimated. In addition, each time the classification is correct, the mean square error between the exact zero location and the estimated ones is computed, wich it will be the second parameter, i.e.:

$$\text{MSE} = \sqrt{\frac{1}{3}\sum_{i=1}^{3} |\hat{z}_i - z_i|^2} \tag{16}$$

where $\hat{z}_i$ are the estimated zeros and z_i the theoretical ones. The resulting MSE is shown in the upper graph in Figure 1 using the correct clasifications in the 20 independent realizations.

With the first parameter it is possible study the probability the method is wrong in the label of a zero as a correct zero, and with the second it is possible to study how accurate the method places the correct zeros. The behaviour of both parameters as a function of the SNR are shown in Figure 1 for the two approaches described in Section 3, specifically, the solution using Levinson algorithm with a determinate system of equations and the standard LS method for an overdeterminate system of equations. This is study for two values of p, specifically, $p = 4$ and $p = 10$.

As it is shown in this figure, the LS method gives a better probability of correct zero location and a lower MSE than the Levinson procedure. It can be observed that increasing the p index of the system of equations (7), i.e. overdetemine the problem, allows the estimations to be improved, but there is a limit in the improvement that can be obtained by this method, since the increasing of p also implies to increase the number of spurious zeros which can be erroneously taken as the correct ones, decreasing the probability of correct classification, and there can be a drop in the probability of zeros identification. This behaviour is observed in Figure 1, where the Levinson method shows a poorer performance for $p = 10$ than for $p = 4$ in the probability of correct clasification at high SNR. As an important conclusion, it can be drawn from the simulations that the LS method achieves a high probability of correct identification with a low MSE for SNR up to 5 dB (see Figure 1 for $p = 10$). Then, it is proved that it is reliable to estimate the cubic phase frequencies using AR modelling. It is worthy also to note that the number of data in the signal $x(n)$ is not big, so application to real situations even in real time of these approach is possible. Of course, better result will be found if the number of data is increased, since the methods are unbias, so they always reach asyntotically the correct solution.

Example 2. In the following example, the possibility of recognizing a cubic phase coupling from four frequencies harmonically related but not coupled is studied. The signal is now given by:

$$x(n) = \sum_{i=1}^{4} A_i^c e^{j(\nu_i^c n + \psi_i^c)} + \sum_{i=1}^{4} A_i^u e^{j(\nu_i^u n + \psi_i^u)} \tag{17}$$

where ν_4^c and ν_1^u verifies that $\nu_4^c = \nu_1^c + \nu_2^c + \nu_3^c$ and $\nu_4^u = \nu_1^u + \nu_2^u + \nu_3^u$, the phases $\{\psi_k^c,\ k = 1,2,3\}$ and ψ_1^u, ψ_2^u, ψ_3^u and ψ_4^u are i.i.d. random variables uniform in the interval $[0, 2\pi)$ and ψ_4^c being $\psi_4^c = \psi_1^c + \psi_2^c + \psi_3^c$. Therefore, this signal contains a cubic phase coupling and other four frequencies which are harmonically related but not coupled. In the simulation the following frequency values have been chosen:

$$\begin{aligned}(\nu_1^c, \nu_2^c, \nu_3^c, \nu_4^c) &= 2\pi(0.08, 0.26, 0.38, 0.72) \text{ (rad/s)} \\ (\nu_1^u, \nu_2^u, \nu_3^u, \nu_4^u) &= 2\pi(0.18, 0.3, 0.48, 0.96) \text{ (rad/s)}\end{aligned} \tag{18}$$

The signal consisted in 64 independent records with 64 data values each one. For this signal, it is neccesary for the records to be

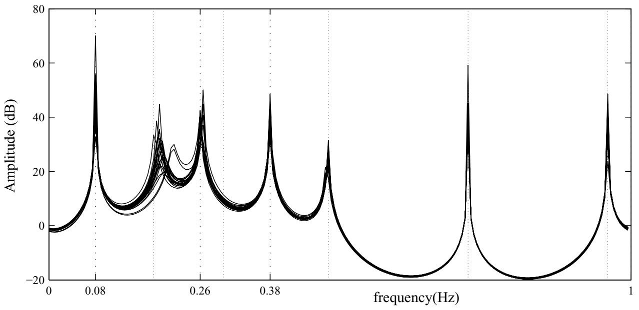

Figure 2: Power spectrum of the AR frequency coefficients obtained by the LS method with p=10 for ten realizations of a signal with a cubic phase coupling and other four harmonically related frequencies.

independent, since the signal is not ergodic. A detail explication of this problem can be found in [1].

After the obtaining of the fourth-order cumulants, the system of equations (7) is built and then solved by the Levinson's algorithm and the LS method, as it was in the first example. In Figures 2 and 3, ten spectrum of the AR coefficients estimated for noiseless signals, with p setting equal to 7 for both methods, are shown. In this figure and in the nexts, the vertical continuous line represents the individual frequency positions, which is where we want to obtain the spectrum peaks, while the vertical dashed lines show the location of the remaining frequencies of the signal (17). The theoretical form of the spectrum of the AR coefficients should be just three peaks in the frequencies correspondent with the individual frequencies of the cubic coupling.

Several conclusions can be drawn from these pictures. The results obtained from the LS method exhibits a lower variance than those given by Levinson solution. The LS method presents clear narrow peaks in the individual frequencies, while the peaks of the Levinson method are less clear and wider. However, the spectrums of the AR coefficients obtained by the LS method present other peaks apart from the expected. Those peaks are also narrow and they appear systematically in the same positions, which correspond with the sum of the individual copled frequencies and the uncoupled frequencies. Those extra peaks are due to the spurious zeros that appear due to the overdetermination of the problem. The fact the spurious zeros form peaks means that they are close to the correct solution, and the fact that they appear systematically in the same position means that this contribution is not possible to be eliminate by any average. As it can be seen in Figure 2, it is not possible to discriminate from the spectrum of the AR coefficients obtained by LS method which frequencies are coupled and which not. In fact, if the number of couplings is unknown, from this figure would indicate that there are two couplings, what would lead to a wrong analysis of the signal.

On the other hand, the spurious zeros of the Levison-base solution produce less peaks, less clear and they do not present the systematicity in the position that the LS's presented. Then, it is easier discriminate in the Levinson method than in the LS the individual coupled frequencies. This is highligthed if a spectral averaging is taken, as it is shown in Figure 4. The resulting averaged spectrum method shows only clear peaks at the individual coupled frequencies, thus allowing the coupled frequencies to be distinguish from those only harmonically related. Once this discrimination is done, the LS results can be used to estimate the individual coupled frequencies with better precision. Then a combination of the two AR method allows us to a correct analysis of the nature of the signal and an acurate location of the individual coupled signals.

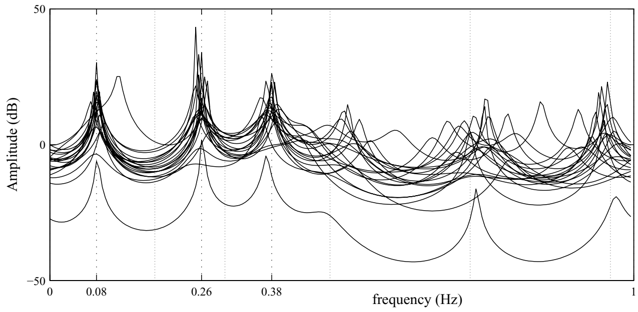

Figure 3: Power spectrum of the AR frequency coefficients obtained by the Levinson method with p=10 for ten realizations of a signal with a cubic phase coupling and other four harmonically related frequencies.

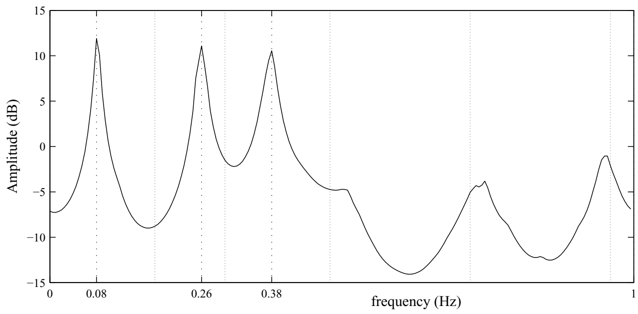

Figure 4: Mean of ten power spectrums of the AR frequency coefficients obtained by the Levinson method with p=10 for a signal with a cubic phase coupling and other four harmonically related frequencies.

5. CONCLUSIONS

In this communication, the problem of detecting cubic phase coupling using AR modelling is studied. It is proposed a procedure based on a double AR modelling and a further detection and selection of coupled frequencies. The proposed method minimizes the computational burden associated to the maximum-detection algorithms in the trispectral domain estimating the frequencies by locating the zeroes of a low-order polynomial. Simulation results show the proposed method allows adequate estimations and the viability of the procedure to distinguish between cubic phase coupling and four frequencies harmonically related but not coupled.

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