AN ENERGY-CONSERVING DIFFERENCE SCHEME FOR NONLINEAR COUPLED TRANSVERSE/LONGITUDINAL STRING VIBRATION

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ABSTRACT

In the quest for more realistic string sound synthesis, nonlinear (large-amplitude) effects have recently come under scrutiny. Though a mathematical description of the coupled longitudinal/transverse motion of such a string is straightforward, the development of numerical methods (and thus synthesis algorithms) is complicated due to stability considerations; frequency domain analysis cannot be fruitfully applied when nonlinearities are present. We present here a finite difference scheme for a nonlinear string whose stability can be guaranteed, not through frequency domain analysis, but through an exact discrete energy conservation property. Under certain simple conditions, the so-called energy method leads to bounds on the solution size in terms of initial conditions, and, thus, to a stability guarantee. Implementation details and numerical results are presented.

1. INTRODUCTION

In recent years, the problem of the numerical simulation of nonlinear (large-amplitude) string vibration for sound synthesis applications has seen increased attention; the timbre of many stringed instruments (such as, e.g., the Finnish kantele [1, 2] and piano [3, 4]) is dependent to a lesser or greater degree on fundamentally nonlinear effects. Whereas digital waveguides [5] offer an admirably efficient alternative to general numerical schemes for the simulation of linear (small-amplitude) string vibration, they do not extend readily to the nonlinear case. Given the increase in computational power over recent years, the solution of such systems by direct numerical means has become a possibility. Finite difference schemes have long been applied to linear string vibration [6, 7], and have been extended, to a limited degree, to certain nonlinear forms [8, 4, 9]. In recent work, this author [10] has discussed finite difference schemes for transverse nonlinear string vibration as modelled by the Kirchhoff-Carrier equation [11]. Such schemes are of energy-conserving type [12], and offer global, energy-based stability conditions that the earlier attempts do not.

The Kirchhoff-Carrier string vibration model, while nonlinear, is unsatisfying in that longitudinal motion is averaged out. In this article, we extend finite difference schemes to deal with a more complete model of nonlinear string vibration for which the longitudinal and transverse coupling remains. As we will show, a properly-chosen difference scheme possesses a conserved discrete energy analogous to the energy conserved by the model system, and leads to a simple global stability condition.

2. A MODEL SYSTEM

A general model of nonlinear string vibration [13, 14], including coupled longitudinal and transverse motion in a single polarization, is given as follows:

$$\rho\ddot{\xi} = EA\xi'' - (EA-T)\left(\frac{1+\xi'}{\sqrt{(1+\xi')^2+(\eta')^2}}\right)' \quad \text{(1a)}$$

$$\rho\ddot{\eta} = EA\eta'' - (EA-T)\left(\frac{\eta'}{\sqrt{(1+\xi')^2+(\eta')^2}}\right)' \quad \text{(1b)}$$

Here, $\xi(x,t)$ and $\eta(x,t)$ describe the deviation of a point on the string as a function of time $t \geq 0$ and distance along the string $x \in [0, L]$. ξ corresponds to longitudinal motion, and η to transverse motion in a single transverse polarization, and note that the two types of motion are coupled by the last terms in both equations. Dots and primes indicate partial differentiation with respect to time and space, respectively. The parameters E, A, ρ and T are Young's modulus, cross-sectional area, linear mass density, and nominal tension for the string, all assumed constant.

Introducing the variables

$$p_\xi = \dot{\xi} \qquad q_\xi = \xi' \qquad p_\eta = \dot{\eta} \qquad q_\eta = \eta'$$

and making use of a series approximation to the nonlinearity, we obtain

$$\rho\dot{p}_\xi = EAq'_\xi + \frac{EA-T}{2}\;q_\eta^2\;{}' \quad \text{(2a)}$$

$$\rho\dot{p}_\eta = Tq'_\eta + \frac{EA-T}{2}\;q_\eta^3 + 2q_\eta q_\xi\;{}' \quad \text{(2b)}$$

$$\dot{q}_\xi = p'_\xi \quad \text{(2c)}$$

$$\dot{q}_\eta = p'_\eta \quad \text{(2d)}$$

We note that this system is slightly different from the third-order system normally seen in the literature [13, 15], due to exclusion of certain terms, based on the fact that ξ is of the same order as η^2 [16, 13]. We will consider two types of boundary conditions: for analysis purposes, we will examine so-called periodic boundary conditions of the form

$$\xi(0,t) = \xi(L,t) \qquad \eta(0,t) = \eta(L,t) \quad \text{(3)}$$

and then those of the fixed type, i.e.,

$$\xi(0,t) = \xi(L,t) = 0 \qquad \eta(0,t) = \eta(L,t) = 0 \quad \text{(4)}$$

2.1 Energetic Analysis

To facilitate the development and analysis of finite difference schemes, we now derive a conserved energy for system (2). Multiplying the first and second equations by p_ξ and p_η respectively, and integrating over the range $x \in [0, L]$ gives

$$\int_0^L \rho p_\xi \dot{p}_\xi dx = \int_0^L p_\xi \quad EAq_\xi + \frac{EA-T}{2}q_\eta^2 \quad{}' \; dx$$

$$\int_0^L \rho p_\eta \dot{p}_\eta dx = \int_0^L p_\eta \quad Tq_\eta + \frac{EA-T}{2}(q_\eta^3 + 2q_\xi q_\eta) \quad{}' \; dx$$

Using integration by parts, boundary conditions (3) or (4), as well as definitions (2c) and (2d), we can write

$$\int_0^L \rho p_\xi \dot{p}_\xi dx = -\int_0^L \dot{q}'_\xi \quad EAq_\xi - \frac{EA-T}{2}q_\eta^2 \quad dx$$
$$\int_0^L \rho p_\eta \dot{p}_\eta dx = -\int_0^L \dot{q}'_\eta \quad Tq_\eta - \frac{EA-T}{2}(q_\eta^3 + 2q_\xi q_\eta) \quad dx$$

We can then conclude that

$$\frac{d}{dt}\mathcal{H} = 0 \qquad \Longrightarrow \qquad \mathcal{H} = \text{constant}$$

for the scalar quantity $\mathcal{H}$ defined by

$$\mathcal{H} = \frac{\rho}{2} \quad \|p_\xi\|^2 + \|p_\eta\|^2 \quad + \frac{T}{2} \quad \|q_\xi\|^2 + \|q_\eta\|^2 + \frac{EA-T}{8}\|q_\eta^2 + 2q_\xi\|^2 \tag{5}$$

In the above expression, we have used the notation $\|f\| = (\int_0^L f^2 dx)^{1/2}$, signifying a spatial 2-norm over $x \in [0, L]$.

$\mathcal{H}$ is non-negative for $EA \geq T$, and under this condition we can then arrive at the useful bounds

$$\|p_\xi\|, \|p_\eta\| \leq \sqrt{\frac{2\mathcal{H}}{\rho}} \qquad \|q_\xi\|, \|q_\eta\| \leq \sqrt{\frac{2\mathcal{H}}{T}} \tag{6}$$

which hold at all times $t \geq 0$. Essentially, the state of the system is bounded in terms of the initial energy $\mathcal{H}$.

3. FINITE DIFFERENCE SCHEMES

In this section, we present some basic facts about grid functions and difference operators, and then turn immediately to a particular finite difference scheme for system (2).

A grid function f_i^n over a bounded domain, as employed by a finite difference scheme, is defined for integers $n \geq 0$ and $i \in [0, \dots, N-1]$ and is an approximation to a function $f(x,t)$ at the coordinates $x = ih$ and $t = nk$. Here, k is the time step, and h is the grid spacing. It is also useful to define grid functions such as $g_{i+1/2}^{n+1/2}$, for the same range of integers i and n; such a grid function takes on values at spatial locations and time instants which are *interleaved* [17] with respect to those at which f_i^n is defined.

The forward time difference operator δ_{t+} is defined by

$$\delta_{t+} f_i^n = \frac{1}{k}(f_i^{n+1} - f_i^n)$$

and forward, backward, and centered time-averaging operators μ_{t+}, μ_{t-} and μ_{t0} are defined by

$$\mu_{t+} f_i^n = \frac{1}{2}(f_i^{n+1} + f_i^n) \qquad \mu_{t-} f_i^n = \frac{1}{2}(f_i^n + f_i^{n-1})$$

$$\mu_{t0} f_i^n = \frac{1}{2}(f_i^{n+1} + f_i^{n-1})$$

Forward and backward spatial difference operators δ_{x+} and δ_{x-} are defined by

$$\delta_{x+} f_i^n = \frac{1}{h}(f_{i+1}^n - f_i^n) \qquad \delta_{x-} f_i^n = \frac{1}{h}(f_i^n - f_{i-1}^n)$$

For periodic boundary conditions, the spatial indices of the grid function are to be taken modulo N. For instance, $\delta_{x+} f_{N-1}^n = \frac{1}{h}(f_0^n - f_{N-1}^n)$. We note that all the operators defined above commute with one another.

The discrete spatial l^2 inner product at time step n between two real-valued grid functions f_i^n and g_i^n, and the associated norm are defined by

$$\langle f^n, g^n \rangle = \sum_{i=0}^{N-1} h f_i^n g_i^n \qquad \|f^n\| = \langle f^n, f^n \rangle^{1/2}$$

The inequality

$$\|\delta_{x-} f^n\| \leq \frac{2}{h}\|f^n\| \tag{7}$$

follows immediately from an application of the Cauchy-Schwartz and triangle inequalities [18].

Integration by parts may be transferred to the discrete case, for periodic boundary conditions, as

$$\langle f^n, \delta_{x-} g^n \rangle = -\langle \delta_{x+} f^n, g^n \rangle \tag{8}$$

One finite difference scheme for system (2) (and there are obviously many such choices) is given by

$$\rho\delta_{t+} p_{\xi,i}^{n-1} = EA\delta_{x-} q_{\xi,i+1/2}^{n-1/2} + \frac{EA-T}{2}\delta_{x-} \quad q_{\eta,i+1/2}^{n-1/2} \mu_{t0} q_{\eta,i+1/2}^{n-1/2} \tag{9a}$$

$$\rho\delta_{t+} p_{\eta,i}^{n-1} = T\delta_{x-} q_{\eta,i+1/2}^{n-1/2} + \frac{EA-T}{2}\delta_{x-} \quad (q_{\eta,i+1/2}^{n-1/2})^2 \mu_{t0} q_{\eta,i+1/2}^{n-1/2} + \frac{EA-T}{2}\delta_{x-} \quad q_{\eta,i+1/2}^{n-1/2} \mu_{t+}\mu_{t-} q_{\xi,i+1/2}^{n-1/2} \tag{9b}$$

$$\delta_{t+} q_{\xi,i+1/2}^{n-1/2} = \delta_{x+} p_{\xi,i}^n \tag{9c}$$

$$\delta_{t+} q_{\eta,i+1/2}^{n-1/2} = \delta_{x+} p_{\eta,i}^n \tag{9d}$$

Notice that it is interleaved, i.e. the grid functions $p_{\xi,i}^n$, $p_{\eta,i}^n$ and $q_{\xi,i+1/2}^{n-1/2}$, $q_{\eta,i+1/2}^{n-1/2}$ are calculated for integer n and i only. It is clearly not the simplest possible such algorithm, in that it is implicit [19], and an implementation will require the solution of a sparse linear system at each time step. It does, however, possess a conserved quantity analogous to an energy which we will examine presently.

3.1 Energetic Analysis

In order to derive a conserved quantity from (9), we carry out steps similar to those performed in Section 2.1. Consider, for the moment, equation (9a). Taking the inner product of both sides of this equation with $\mu_{t+} p_i^{n-1}$ gives

$$0 = \delta_{t+}\frac{\rho}{2}\|p_\xi^{n-1}\|^2 - EA\langle \mu_{t+} p_\xi^{n-1}, \delta_{x-} q_\xi^{n-1/2} \rangle - \frac{EA-T}{2}\langle \mu_{t+} p_\xi^{n-1}, \delta_{x-} \quad q_\eta^{n-1/2}\mu_{t0} q_\eta^{n-1/2} \quad \rangle$$

where we have used the fact that $\langle \mu_{t+} p_\xi^{n-1}, \delta_{t+} p_\xi^{n-1} \rangle = \delta_{t+}\|p_\xi^{n-1}\|^2/2$. Continuing, we have

$$\begin{aligned}
0 &= \delta_{t+}\frac{\rho}{2}\|p_\xi^{n-1}\|^2 + EA\langle \mu_{t+}\delta_{x+} p_\xi^{n-1}, q_\xi^{n-1/2} \rangle + \frac{EA-T}{2}\langle \mu_{t+}\delta_{x+} p_\xi^{n-1}, q_\eta^{n-1/2}\mu_{t0} q_\eta^{n-1/2} \rangle \\
&= \delta_{t+}\frac{\rho}{2}\|p_\xi^{n-1}\|^2 + EA\langle \mu_{t+}\delta_{t+} q_\xi^{n-3/2}, q_\xi^{n-1/2} \rangle + \frac{EA-T}{2}\langle \mu_{t+}\delta_{t+} q_\xi^{n-3/2}, q_\eta^{n-1/2}\mu_{t0} q_\eta^{n-1/2} \rangle \\
&= \delta_{t+} \quad \frac{\rho}{2}\|p_\xi^{n-1}\|^2 + \frac{EA}{2}\langle q_\xi^{n-3/2}, q_\xi^{n-1/2} \rangle \qquad (10) \\
&\quad + \frac{EA-T}{2}\langle \mu_{t+}\delta_{t+} q_\xi^{n-3/2}, q_\eta^{n-1/2}\mu_{t0} q_\eta^{n-1/2} \rangle
\end{aligned}$$

where in first two steps above we have used integration by parts (8) and commutativity of the operators μ_{t+} and δ_{x+}, definition (9c), respectively.

After a similar series of steps, applied to the inner product of (9b) with $\mu_{t+}p^n_{\eta,i}$, we arrive at

$$
\begin{aligned}
0 \;=\; & \delta_{t+}\left(\frac{\rho}{2}\|p_\eta^{n-1}\|^2 + \frac{T}{2}\langle q_\eta^{n-3/2}, q_\eta^{n-1/2}\rangle\right) \\
& + \frac{EA-T}{2}\langle \mu_{t+}\delta_{t+}q_\eta^{n-3/2}, (q_\eta^{n-1/2})^2\mu_{t0}q_\eta^{n-1/2}\rangle \\
& + \frac{EA-T}{2}\langle \mu_{t+}\delta_{t+}q_\eta^{n-3/2}, q_\eta^{n-1/2}\mu_{t+}\mu_{t-}q_\xi^{n-1/2}\rangle
\end{aligned}
\tag{11}
$$

Taking the sum of (10) and (11) above, and performing additional manipulations, we arrive, finally, at

$$
\delta_{t+}\mathcal{H}^{n-1} = 0 \qquad \Longrightarrow \qquad \mathcal{H}^n = \text{constant}
$$

where $\mathcal{H}^n$ is defined by

$$
\begin{aligned}
\mathcal{H}^n \;=\; & \frac{\rho}{2}\left(\|p_\xi^n\|^2 + \|p_\eta^n\|^2\right) \\
& + \frac{EA}{2}\langle q_\xi^{n+1/2}, q_\xi^{n-1/2}\rangle + \frac{T}{2}\langle q_\eta^{n+1/2}, q_\eta^{n-1/2}\rangle \\
& + \frac{EA-T}{8}\left(\|q_\eta^{n+1/2}q_\eta^{n-1/2} + 2\mu_{t+}q_\xi^{n-1/2}\|^2 \right. \\
& \left. \qquad -4\|\mu_{t+}q_\xi^{n-1/2}\|^2\right)
\end{aligned}
$$

$\mathcal{H}^n$ is thus a *conserved quantity* for scheme (9), and can be thought of as a discrete counterpart to the energy $\mathcal{H}$ for the model system (2) as defined by (5).

3.2 Numerical Stability

A conserved energy-like quantity in a difference scheme is not enough to show numerical stability; we must also find the conditions under which it is positive for all possible choices of the state. To this end, we introduce the variables $\tilde{q}^n_{\xi,i+1/2} = \mu_{t+}q^{n-1/2}_{\xi,i+1/2}$ and $\tilde{q}^n_{\eta,i+1/2} = \mu_{t+}q^{n-1/2}_{\eta,i+1/2}$. $\mathcal{H}^n$ can then be rewritten as

$$
\begin{aligned}
\mathcal{H}^n \;=\; & \frac{1}{2}\left(\rho\|p_\xi^n\|^2 - \frac{EAk^2}{4}\|\delta_{x+}p_\xi^n\|^2 + T\|\tilde{q}_\xi^n\|^2\right) \\
& + \frac{1}{2}\left(\rho\|p_\eta^n\|^2 - \frac{Tk^2}{4}\|\delta_{x+}p_\eta^n\|^2 + T\|\tilde{q}_\eta^n\|^2\right) \\
& + \frac{EA-T}{8}\|(\tilde{q}_\eta^n)^2 + 2\tilde{q}_\xi^n - \frac{k^2}{4}(\delta_{x+}p_\eta^n)^2\|^2
\end{aligned}
$$

Given (8), we may then write

$$
\begin{aligned}
\mathcal{H}^n \;\geq\; & \frac{1}{2}(\rho - EA\alpha^2)\|p_\xi^n\|^2 + \frac{T}{2}\|\tilde{q}_\xi^n\|^2 \\
& + \frac{1}{2}(\rho - T\alpha^2)\|p_\eta^n\|^2 + \frac{T}{2}\|\tilde{q}_\eta^n\|^2 \\
& + \frac{EA-T}{8}\|(\tilde{q}_\eta^n)^2 + 2\tilde{q}_\xi^n - \frac{k^2}{4}(\delta_{x+}p_\eta^n)^2\|^2
\end{aligned}
$$

where we have defined $\alpha = k/h$. Under the conditions

$$
\alpha \leq \sqrt{\frac{\rho}{T}}, \sqrt{\frac{\rho}{EA}} \qquad\qquad EA \geq T \tag{12}
$$

then all five terms in the above expression for $\mathcal{H}^n$ are non-negative, and we have $\mathcal{H}^n \geq 0$, for any choices of the state variables. (The conditions above on α, the time-step/space-step ratio, should be familiar as the Courant-Friedrichs-Lewy condition [19].) Under these conditions, we can conclude that

$$
\|p_\xi^n\| \leq \sqrt{\frac{2\mathcal{H}^n}{\rho - EA\alpha^2}} \qquad \|p_\eta^n\| \leq \sqrt{\frac{2\mathcal{H}^n}{\rho - T\alpha^2}} \tag{13}
$$

and also, from the relationship between $\tilde{q}^n_{\xi,i+1/2}$, $\tilde{q}^n_{\eta,i+1/2}$ and $q^{n-1/2}_{\xi,i+1/2}$,$q^{n-1/2}_{\xi,i+1/2}$, that

$$
\|q_\xi^n\| \;\leq\; \sqrt{\frac{2\mathcal{H}^n}{T}}\left(1 + \alpha\sqrt{\frac{T}{\rho - EA\alpha^2}}\right) \tag{14a}
$$

$$
\|q_\eta^n\| \;\leq\; \sqrt{\frac{2\mathcal{H}^n}{T}}\left(1 + \alpha\sqrt{\frac{T}{\rho - T\alpha^2}}\right) \tag{14b}
$$

Bounds (13) and (14) are our guarantee of numerical stability for scheme (9), when conditions (12) are satisfied.

3.3 Implementation Details

For the sake of simplicity, the results in the previous sections were derived under the assumption of periodic boundary conditions, as given by (3). We have shown, in previous work [10] that the results also hold in the case of fixed boundary conditions, as specified by (4). In this case, under the assumptions $p^n_{\xi,0} = p^n_{\eta,0} = p^n_{\xi,N} = p^n_{\eta,N} = 0$, and writing the vectors

$$
\begin{aligned}
\mathbf{p}_\xi^n &= [p^n_{\xi,1}, \ldots, p^n_{\xi,N-1}]^T \\
\mathbf{p}_\eta^n &= [p^n_{\eta,1}, \ldots, p^n_{\eta,N-1}]^T \\
\mathbf{q}_\xi^{n-1/2} &= [q^{n-1/2}_{\xi,1/2}, \ldots, q^{n-1/2}_{\xi,N-1/2}]^T \\
\mathbf{q}_\eta^{n-1/2} &= [q^{n-1/2}_{\eta,1/2}, \ldots, q^{n-1/2}_{\eta,N-1/2}]^T
\end{aligned}
$$

scheme (2) may be written as

$$
\begin{bmatrix}\mathbf{p}_\xi^n \\ \mathbf{p}_\eta^n\end{bmatrix} = \begin{bmatrix}\mathbf{p}_\xi^{n-1} \\ \mathbf{p}_\eta^{n-1}\end{bmatrix} + 2\frac{\rho}{EA-T}(\mathbf{A}^{n-1/2})^{-1}\begin{bmatrix}\mathbf{D}_- & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_-\end{bmatrix}\mathbf{f}^{n-1/2}
$$

$$
\begin{bmatrix}\mathbf{q}_\xi^{n+1/2} \\ \mathbf{q}_\eta^{n+1/2}\end{bmatrix} = \begin{bmatrix}\mathbf{q}_\xi^{n-1/2} \\ \mathbf{q}_\eta^{n-1/2}\end{bmatrix} + 2\frac{\rho}{EA-T}\begin{bmatrix}\mathbf{D}_+ & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_+\end{bmatrix}\begin{bmatrix}\mathbf{p}_\xi^n \\ \mathbf{p}_\eta^n\end{bmatrix}
$$

where

$$
\mathbf{f}^{n-1/2} = \begin{bmatrix} EA\mathbf{q}_\xi^{n-1/2} + \frac{EA-T}{2}(\mathbf{q}_\eta^{n-1/2})^{\star 2} \\ T\mathbf{q}_\eta^{n-1/2} + \frac{EA-T}{2}((\mathbf{q}_\eta^{n-1/2})^{\star 3} + 2\mathbf{q}_\xi^{n-1/2} \star \mathbf{q}_\eta^{n-1/2}) \end{bmatrix}
$$

$$
\mathbf{A}^{n-1/2} = \begin{bmatrix} \mathbf{I} & -\mathbf{D}_-\mathrm{diag}(\mathbf{q}_\eta^{n-1/2})\mathbf{D}_+ \\ -\mathbf{D}_-\mathrm{diag}(\mathbf{q}_\eta^{n-1/2})\mathbf{D}_+ & \mathbf{I} - \mathbf{D}_-\mathrm{diag}(\mathbf{q}_\eta^{n-1/2})^{\star 2}\mathbf{D}_+ \end{bmatrix}
$$

$$
\mathbf{D}_+ = \frac{\alpha}{2}\sqrt{\frac{EA-T}{\rho}}\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix} \qquad \mathbf{D}_- = -\mathbf{D}_+^T
$$

where the $\star$ indicates element-by-element vector multiplication or exponentiation. It is worth noting that although scheme (9) is implicit, it may be solved uniquely at every time-step. In addition, a direct inversion of $\mathbf{A}^{n-1/2}$ is unnecessary, it suffices to solve a linear system.

4. NUMERICAL RESULTS

As a simple test of this numerical method, we consider a string of length 0.65 m, made of steel (of linear density $\rho = 6 \times 10^{-4}$ kg/m and with Young's Modulus $E = 2 \times 10^{11}\text{N/m}^2$), of cross-sectional area $A = 3.6 \times 10^{-8}\text{m}^2$, and under tension $T = 120$N. For accurate simulation results, a sample rate of 1 MHz is used. We subject the string to struck conditions, with an initial velocity distribution of the form of a raised cosine centered at the string center—snapshots of the time evolution of the string profile are shown, across the rows, in Figure 1, for three different strike velocities, 10 m/s, 500 m/s and 1000 m/s. In the first case, the string is essentially linear, while for the higher velocities, the nonlinear behaviour is evident, leading in particular to harmonic generation and an increase in the propagation speed of the disturbance. In all three cases, energy is conserved to machine arithmetic for the duration of the simulation.

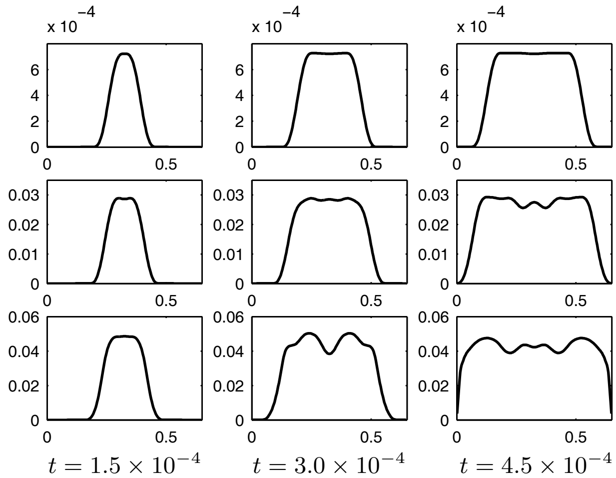

Figure 1: Time evolution (vertically across the rows) of the profile of a string described by system (2), under the application of difference scheme (9), for a variety of velocities (down the columns). Distances are given in metres on both axes.

5. CONCLUDING REMARKS

The primary result in this paper is the proof of stability of a finite difference scheme for a general nonlinear string system; we reiterate that frequency domain ideas have not been applied here; stability is proved via energetic analysis. This would appear to be a substantial benefit, especially in the area of musical sound synthesis. We do note, however, that such analysis is a delicate tool—as we have seen, in order for it to be applicable, we require (a) that the model system itself possess a positive energy function, (b) that the difference scheme be conservative, and (c) that the discrete conserved energetic quantity be positive; these conditions are frequently not met by schemes used in practice. One response to these observations is that such a technique is overly restrictive; another is that perhaps it would do well to follow the "suggestions" that an energetic viewpoint offers.

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