CLOSED-FORM DESIGN OF DIGITAL FRACTIONAL-DELAY FILTERS WITH CONSTANT MAGNITUDE AND VARIABLE DELAY

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ABSTRACT

In most approximation techniques for implementing a variable fractional delay, like Lagrange interpolation, the magnitude response varies considerably with the delay. Instead, it would be desirable to keep the magnitude response the same for all delay values. In this paper, we propose a novel method for optimizing the parameters of the least-squared error spline transition band fractional delay FIR design to achieve good delay approximation with delay-independent magnitude response.

1. INTRODUCTION

In many applications, a variable fractional-delay (FD) filter is needed [1]. Typically the magnitude response of FIR FD filters also varies with the chosen delay value, which is usually undesirable. One alternative is to use allpass IIR FD filters that have unity magnitude response at all frequencies, but unfortunately they suffer from transient effects whenever the delay value is adjusted on-line [6]. For FIR filters, instead, the simple Lagrange interpolator has excellent approximation at low frequencies for both the magnitude and delay, but the magnitude response outside the passband varies strongly with the delay value. Even if this would not affect the signal being delayed, it might cause harmful variations in the system noise characteristics.

The least-squared error (LSE) spline transition-band FD filter [2], [3], instead, has been observed to have a magnitude response that remains almost unchanged over different fractional delays [1], [4], (Fig. 1). The cost is a slightly more narrow passband. The choice of the design parameters (filter length N, spline order P, and passband and stopband edge frequencies) were discussed in detail in [2] for conventional linear-phase lowpass FIR filter design. However, the design of FD filters has important differences. Usually, no explicit stopband region is needed. Furthermore, the filter length is typically rather small (ca. 4-20) and being the key design parameter it is fixed first, whereas the passband edge frequency is not so critical and can be chosen as resources permit. Last but not least, the (phase) delay response should also meet requirements, which is of course not an issue with linear-phase FIR design.

In this paper, we derive design equations for the least-squared error spline transition-band FIR FD filter that attains both good FD phase approximation and nearly fixed magnitude response over all the fractional delay values $D \in [-0.5, 0.5)$. In Section II, we introduce the LSE spline FIR FD design equations. In Section III, the optimization of the passband edge frequency parameter is addressed. The choice of the spline parameter is then discussed in Section IV. In Section V, the results of Sections III and IV are generalized for both odd and even-order filters, and Section VI gives an example case of how the results of this paper can be used in a practical situation. Finally, Section VII concludes the paper.

2. DESIGN EQUATIONS

The magnitude response of the LSE spline transition band FIR filter is defined by [2]

$$A_{\mathrm{id}}(\omega) = \begin{cases} 1, & \omega \in (0, \omega_p] \\ 1 - \frac{1}{2}\left(\frac{\omega_s - \omega_p}{2}\right)^{-P}(\omega - \omega_p)^P, & \omega \in \left(\omega_p, \frac{\omega_p + \omega_s}{2}\right] \\ \frac{1}{2}(-1)^P\left(\frac{\omega_s - \omega_p}{2}\right)^{-P}(\omega - \omega_s)^P, & \omega \in \left(\omega_p, \frac{\omega_p + \omega_s}{2}\right] \end{cases} \quad (1)$$

where $A_{\mathrm{id}}(\omega)$ is even and periodic with period 2π, ω is the normalized angular frequency, the design parameters ω_p and ω_s indicate the passband and the stopband of the filter, and P is the order of the spline.

The impulse response of an ideal FD filter with delay D is obtained by the inverse discrete Fourier transform of $A_{\mathrm{id}}(\omega) \cdot e^{-j\omega D}$:

$$a_D(n) = \left\{ \frac{\sin[\frac{n-D}{2P}(\omega_s - \omega_p)]}{\frac{n-D}{2P}(\omega_s - \omega_p)} \right\}^P \frac{\sin[\frac{n-D}{2}(\omega_s + \omega_p)]}{\frac{n-D}{2P}(\omega_s - \omega_p)} \quad (2)$$

where $n \in (-\infty, \infty)$. In the following, we will set $\omega_s = \pi$, i.e., no stopband is needed. The LSE approximation of the infinitely-long $a_D(n)$ can be achieved by simply truncating it in the range $-L \le n \le L$, and the frequency response of the truncated odd-length non-causal FIR filter with delay D is obtained as

$$H_D(\omega) = \sum_{n=-L}^{n=L} \left[a_D(n) e^{-j\omega n}\right], \quad D \in [-0.5, 0.5) \quad (3)$$

If we define the amplitude response of the truncated filter with $D = 0$ as the desired *zero-phase* amplitude response, the ideal frequency response for a finite-length FD filter with delay D is

$$H_{\mathrm{id},D}(\omega) = A_0(\omega) \cdot e^{-j\omega D} \quad (4)$$

Note that $A_0(\omega)$ is the real-valued magnitude response of $H_D(\omega)|_{D=0}$ and differs from Eq. (1) due to truncation errors.

The truncation in Eq. (3) can be interpreted as convolution in the frequency domain [5]:

$$H_D(\ ) = [A_{\rm id}(\ )\cdot e^{-j\ D}]\otimes W(\ ) \quad (5)$$

$$= \frac{1}{2}\int_{-} A_{\rm id}(\ -\ )e^{-j(\ -\ )D}W(\ )d\ \quad (6)$$

$$= K_D(\ )\cdot e^{-j\ D} \quad (7)$$

where $\otimes$ denotes convolution and $W(\ )$ is the frequency response of an odd-length rectangular truncation window [3]:

$$W(\ ) = \sum_{n=-L}^{L} e^{-j\ n} = \frac{\sin\left[\left(L+\frac{1}{2}\right)\ \right]}{\sin\left(\frac{1}{2}\ \right)} \quad (8)$$

and

$$K_D(\ ) = K_{D,{\rm Re}}(\ ) + j\cdot K_{D,{\rm Im}}(\ ) \quad (9)$$

of Eq. (7) can be seen to approximate $A_0(\ )$ in Eq. (4). The difference between $A_0(\ )$ and $K_D(\ )$ causes variation of magnitude and phase response between the chosen delay values. Let us define the complex-valued error as:

$$E(\ ,D) = A_0(\ ) - K_D(\ )$$

$$= [A_0(\ ) - K_{D,{\rm Re}}(\ )] - j\cdot K_{D,{\rm Im}}(\ ) \quad (10)$$

which is the deviation of $H_D(\ )$ from the desired zero-phase response $A_0(\ )$. The absolute value of $E(\ ,D)$ is a measure of deviation between the resulting magnitude response and the desired one, while the imaginary part of $E(\ ,D)$ should be set to zero to avoid phase delay error. As a matter of fact, $E(\ ,D)$ varies with both and D, and we use the peak passband error magnitude as an overall design criterion to simplify the problem:

$$C = \max\left\{|E(\ ,D)|,\ \ \in(0,\ _p), D\in(-0.5,0.5)\right\} \quad (11)$$

Our objective is to minimize C in the desired frequency range $\in(0,\ _p)$, which means making $K_D(\ )$ as close to $A_0(\ )$ as possible and minimizing the variation of magnitude and phase response with the chosen delay values. As can be seen from Figs. 2 and 3, C varies substantially with respect to $_p$ and P, and there seem to be obtimal choices for these parameters that minimize C. In the following two sections, we propose methods to find the best values for the parameters $_p$ and P.

3. OPTIMAL CHOICE FOR $_P$

The integral in Eq. (6) is so complex that it is difficult to obtain an explicit design equation for the chosen optimization parameters. However, we can find an approximate solution by employing the properties of $A_{\rm id}(\ )$ and $W(\ )$ from equations (1) and (8), respectively.

A key point in the optimization is that ideally the imaginary part of $E(\ ,D)$, which is equal to $-K_{D,{\rm Im}}(\ )$, should be zero, but due to the FD approximation (truncation) a nonzero imaginary component is introduced. We now proceed to characterize an approximate solution by setting the imaginary part $K_{D,{\rm Im}}(\ )$ to zero for $D = 0.5$, which usually has the largest group delay error:

$$K_{(0.5),{\rm Im}}(\ ) = \frac{1}{2}\int_{-} A_{\rm id}(\ -\ )W(\ )\sin(D\ )d\ \Big|_{D=0.5}$$

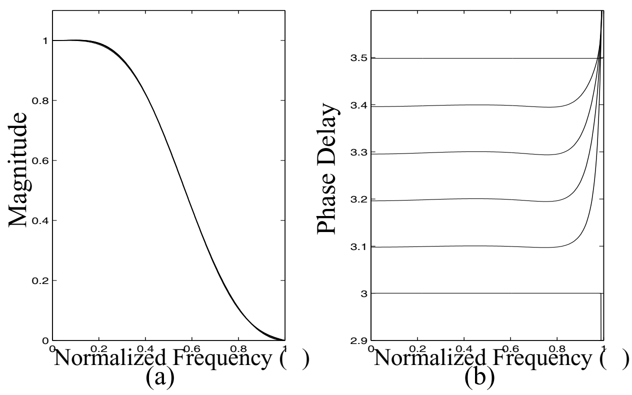

Figure 1: (a) Magnitude and (b) Phase Delay response of 7-tap FD filters, using optimal $_p = 0.14$, and $P = 2$ with $D = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. Compare with the non-optimized design shown in Fig. 5.

$$= \frac{1}{2}\int_{-} A_{\rm id}(\ -\ )\sin\left[\left(L+\frac{1}{2}\right)\ \right]d\ \quad (12)$$

Note that $\sin\left[\left(L+\frac{1}{2}\right)\ \right]$ in Eq. (12) is an odd function, and the integral of such an odd function in the range $\in(-\ ,\ )$ is equal to 0. It is possible to separate $A_{\rm id}(\ -\ )$ into two parts, one a symmetric function of , another an anti-symmetric function of . To do this, we define an auxiliary even periodic function $F(\ ) = F(-\ )$, where

$$F(\ ) = \begin{cases} A_{\rm id}(\ -\ ), & \in(-\ +\ ,0) \\ A_{\rm id}(\ +\ ), & \in(0,-\ -\ ) \\ 0, & \in(-\ ,-\ +\ )\bigcup(\ -\ ,\ ) \end{cases} \quad (13)$$

whose period is 2 . We can replace $A_{\rm id}(\ -\ )$ in Eq. (12) by $\{F(\ ) + [A_{\rm id}(\ -\ ) - F(\ )]\}$ and rewrite it as

$$K_{(0.5),{\rm Im}}(\ ) = \frac{1}{2}\int_{-} F(\ )\sin\left[\left(L+\frac{1}{2}\right)\ \right]d$$

$$+\frac{1}{2}\int_{-} [A_{\rm id}(\ -\ ) - F(\ )]\sin\left[\left(L+\frac{1}{2}\right)\ \right]d\ \quad (14)$$

Because the product of $F(\ )$ and $\sin\left[\left(L+\frac{1}{2}\right)\cdot\ \right]$ is anti-symmetric, the first integral is zero. Furthermore, the second integral in Eq. (14) is a periodic convolution so that the integration range can be shifted by any amount, e.g., . The problem now reduces to minimizing the integral

$$K_{(0.5),{\rm Im}}(\ )$$

$$= \frac{1}{2}\int_{-\ +}^{\ +} [A_{\rm id}(\ -\ ) - F(\ )]\sin\left[\left(L+\frac{1}{2}\right)\ \right]d\ \quad (15)$$

in which for any and P, $[A_{\rm id}(\ -\ ) - F(\ )]$ is non-zero only in the range $\in(\ _p -\ ,\ +\ )$ and symmetric with respect to $_0 = [(\ _p -\ ) + (\ -\ )]/2 = (\ _p +\ )$.

The integral of an anti-symmetric function in a symmetric range is equal to zero and thus we can set the integral in Eq. (14) to zero by forcing $\sin\left[\left(L+\frac{1}{2}\right)\cdot\ \right]$ to be anti-symmetric with respect to $_0$. This is equivalent to requiring that

$$\left(L+\frac{1}{2}\right)\cdot\frac{(\ _p +\ )}{2} = K\ ,\ K\in 0,1,2,\cdots \quad (16)$$

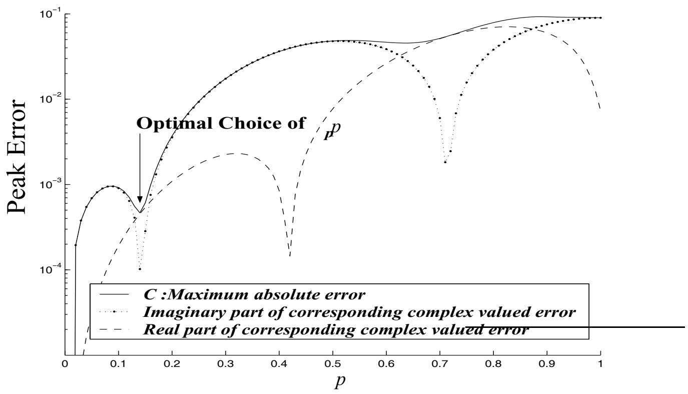

Figure 2: Peak passband error C vs $_p$ with $P = 3, L = 3$

or in terms of $_p$

$$_p = \frac{2K}{L+\frac{1}{2}} - \quad, K = 0,1,2,\cdots \tag{17}$$

which means that for these values of $_p$ the integral in Eq. (12) reduces to zero. Now, let us define the maximum imaginary error as

$$M_{\text{Im}}(\ ) = \max\{|K_{(0.5),\text{Im}}(\ )|\}, \quad \in (0, \ _p) \tag{18}$$

Fig. 2 shows $M_{\text{Im}}(\ )$ with respect to $_p$ (dotted line). It demonstrates the desired result that for the proposed choices of $_p$, the peak value of $K_{(0.5),\text{Im}}(\ )$ is zero in the whole passband. It is also seen that the obtained suboptimum solution can be expected to be quite close to the true optimum when the most narrowband solution for $_p$ is chosen.

4. OPTIMAL CHOICE FOR SPLINE ORDER P

In [2], Burrus et al. suggested a choice for the spline order P resulting in the minimum least-squared error as

$$P_{opt} = \ \cdot (N-1)/(4\ ) \tag{19}$$

where $= \ _p - \ _s$ is the transition band. Because we have no stopband ($_s = \ $), the expression can be elaborated for the integer valued P as

$$\begin{aligned} P_{int} &= \text{round}[(\ - \ _p)\cdot(N-1)/(4\ )] \\ &\approx \text{round}[(N-1)/4] = \text{round}(L/2) \end{aligned} \tag{20}$$

The approximation holds for a relatively narrow passband, which usually is the case in FD filter design. Furthermore, as the typical spline orders are below 5, the quantization to integer orders means that the formula is quite accurate for practical purposes. This has also been observed experimentally, as shown in Fig. 3.

5. GENERALIZATIONS

In the two previous sections, all the optimization was done for the specific case where N is odd and $D = 0.5$. The method can, however, be generalized for even N and other values of D. For $D \neq 0.5$, we have Eq. (12) in the form

$$K_{D,\text{Im}}(\ ) = \frac{1}{2}\int_{-} A_{\text{id}}(\ -\ )\cdot W(\ )\cdot \sin(D\ )d$$

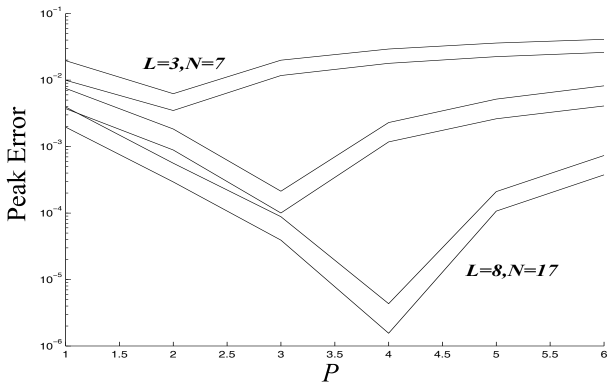

Figure 3: Peak passband error C vs spline order P for odd filter lengths $N = 7, 9, \cdots, 17$.

$$= \frac{1}{2}\int_{-} A_{\text{id}}(\ -\ )\cdot \sin\left[\left(L+\frac{1}{2}\right)\ \right]\frac{\sin(D\ )}{\sin[(1/2)\ ]}d \tag{21}$$

which replaces the constant-amplitude sinusoid function $\sin\left[\left(L+\frac{1}{2}\right)\ \right]$ in Eq. (12) by an amplitude-variant function $\sin\left[\left(L+\frac{1}{2}\right)\ \right]\frac{\sin(D\ )}{\sin[(1/2)\ ]}$ which has similar properties, so that Eq. (16) is still a good choice for $_p$. We have found experimentally that Eq. (20) for P is also a good choice in this case.

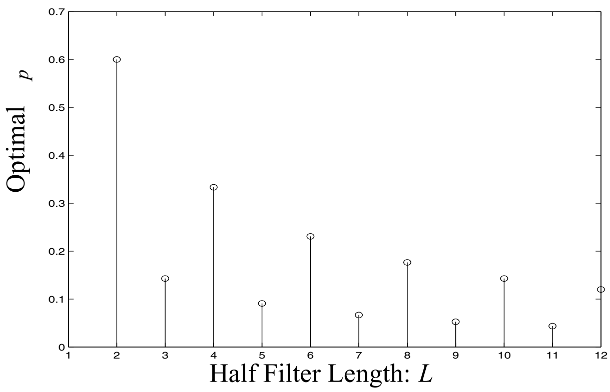

Figure 4: Relationship between L and optimal $_p$

Using the values of $_p$ and P as defined by equations (17) and (20), an example of the resulting filter magnitude and phase delay responses is shown in Fig. 1 for different values of D. In Fig. 1, the filter's magnitude response for 5 different fractional delays D remains almost fixed, and also the phase delay is close to the chosen fractional delay (plus an intrinsic integer delay of the FIR filter). The result is seen to be much better than that of a non-optimized 7-tap filter in Fig. 5, where the magnitude responses are obviously different from each other and the phase response curves suffer from severe ripples. In Fig. 1 the magnitude response has only 1% maximum error for $D = 0.5$ in comparison to $D = 0$ ($L = 3$). A disadvantage of the method is that in Eq. (17) the optimal value for $_p$ has a trend of getting smaller when L increases, as shown in Fig. 4. Hence, the method is best suited for relatively short filters ($L < 10$ or $N < 21$), which is sufficient for most FD applications.

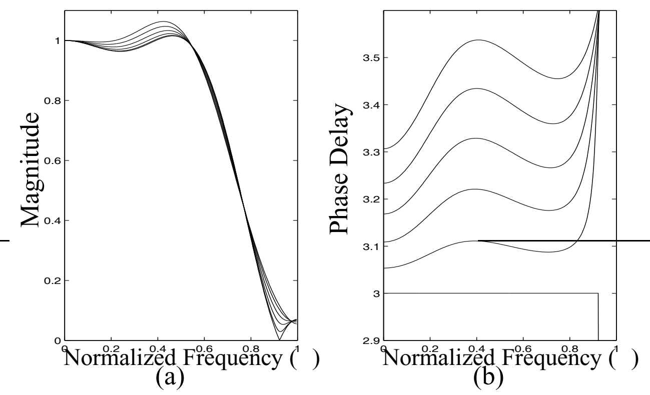

Figure 5: (a) Magnitude and (b) Phase Delay response of non-optimized 7-tap FD filters, using $_p = 0.4$ and $P = 3$ with $D = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. Compare with the optimized design shown in Fig. 1.

Similar derivations for even N yield expressions for P and $_p$ that are the same as those for odd N. For the convenience of the reader, a complete set of design formulas is given in Table 1.

6. DESIGN EXAMPLE

The remarkable improvement that the proposed parameter choices introduce were already illustrated in figures 1 and 5 for an odd filter length (N=7). Let us then consider an even-length filter (N=10). Fig. 6 shows the magnitude and phase delay responses of FD filters with optimal parameter choices $_p = 0.33$ and $P = 2$. The design results show that the magnitude response is almost constant for different values of D. There are 5 magnitude response plots (for different D) in the figure, but they are seen to be almost exactly the same. Fig. 7 illustrates the responses for non-optimal choices $_p = 0.30$ and $P = 6$, which demonstrates slightly worse magnitude behaviour but much deteriorated phase delay approximation.

7. CONCLUSION

We have proposed a method for designing FD filters with flat phase delay response and small magnitude variation. The imaginary and real parts of the error were analyzed and then minimized separately by optimally setting the design parameters. The optimal design parameters are expressed in an explicit form which only depends on the filter length. The formulas enable the design of variable FD filters which have practically constant magnitude response independent of the variable delay value.

Table 1: Summary of Design Formulas

N	$N = 2L+1$(Odd)	$N = 2L+2$(Even)
$_p$	$_p = \frac{2K}{L+\frac{1}{2}} -$	$_p = \frac{2K}{L+\frac{1}{2}} -$
P	$P_{int} = \text{round}(L/2)$	$P_{int} = \text{round}(L/2)$

Restrictions $_p \in (0, \ )$, $2K < L+2$ and $K = 0, 1, 2, 3...$

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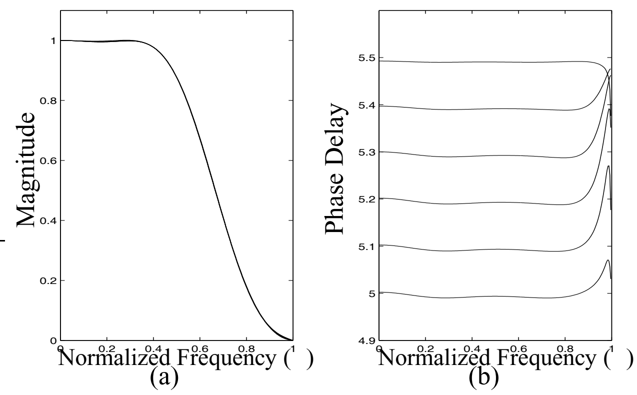

Figure 6: (a) Magnitude and (b) Phase Delay Response of 10-tap FD filters, using optimized $_p = 0.3$ and $P = 6$ with $D = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. Compare with the non-optimazed design shown in Fig. 7.

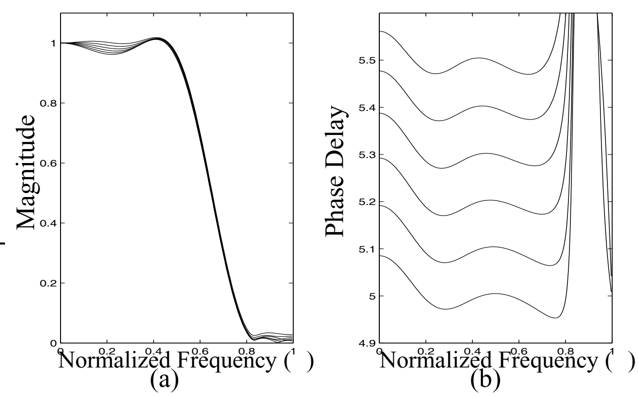

Figure 7: (a) Magnitude and (b) Phase Delay Response of non-optimized 10-tap FD filters, using $_p = 0.33$ and $P = 2$ with $D = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. Compare with the optimazed design shown in Fig. 6.

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