TRACKING ANALYSIS OF VARIABLE XE-NLMF ALGORITHM IN THE PRESENCE OF BOTH RANDOM AND CYCLIC NONSTATIONARITIES

Muhammad Moinuddin and Azzedine Zerguine

Electrical Engineering Department
King Fahd University of Petroleum and Minerals,
Dhahran,31261, Saudi Arabia.
{moinudin, azzedine}@kfupm.edu.sa

ABSTRACT

In this work, tracking analysis of variable normalized least mean fourth (XE-NLMF) algorithm is carried out in the presence of two sources of nonstationarites: 1) carrier frequency offset between transmitter and receiver and 2) random variations in the environment. A novel approach to this analysis is carried out here using the concept of energy conservation. Close agreement between analytical analysis and simulation results is obtained. The results show that, unlike in the stationary case, the steady-state excess mean square error (MSE) is not a monotonically increasing function of the step size.

1. INTRODUCTION

Cyclic and random system nonstationarities are a common impairment in communication systems and especially in applications that involve channel estimation, channel equalization, and inter-symbol-interference cancellation. Random nonstationarity is present due to variations in channel characteristics which is true in most of cases, particularly in the case of a mobile communication environment [1]. Cyclic system nonstationarities arise in communication systems due to mismatches between the transmitter and receiver carrier generator.

The ability of adaptive filtering algorithms to track such system variations is not yet fully understood. In this regard, a recent contribution [2] presented a first-order analysis of the performance of the Least Mean Squares (LMS) algorithm [3] in the presence of the carrier frequency offset. In [4]-[5], a general framework for the tracking analysis of adaptive algorithms was developed. It can handle both cyclic as well as random system nonstationarities simultaneously. This framework, based on an energy conservation principle [6], holds for all adaptive algorithms whose recursions are of the form:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \gamma_{xe}\mathbf{x}_n f(e_n), \tag{1}$$

where $f(e_n)$ denotes a general scalar function of the output estimation error e_n, $\mathbf{w}_n$ is the filter coefficient vector of the adaptive filter, $\mathbf{x}_n$ is the input vector, and γ_{xe} is the step-size used in the adaptation of filter coefficients. In the case of variable normalized least mean fourth (XE-NLMF) algorithm [7], the nonlinearity $f(e_n)$ is defined by:

$$f(e_n) = \frac{e_n^3}{\delta + (1-\alpha_n)\|\mathbf{x}_n\|^2 + \alpha_n\|\mathbf{e}_n\|^2}, \tag{2}$$

where $\|\mathbf{x}_n\|^2$, $\|\mathbf{e}_n\|^2$ are the Eucledian norms of the input sequence $\mathbf{x}_n$ and error vector $\mathbf{e}_n$, respectively. As shown in equation (6), the LMF algorithm [8]-[9] is normalized [10] by both the signal power and error power.

The mixing power parameter, α_n, is confined to the interval [0,1] and will be recursively adapted to adjust the signal power, $\|\mathbf{x}_n\|^2$, and error power, $\|\mathbf{e}_n\|^2$, for maximum performance and is given by [7]:

$$\alpha_n = \frac{2}{\sqrt{\pi}}\int_0^{\mu_n} e^{-y^2}dy, \tag{3}$$

with μ_n updated according to:

$$\mu_{n+1} = \nu\mu_n + p_n|e_n e_{n-1}|. \tag{4}$$

The quantity p_n is updated according to the weighted sum of the past three samples of α_n in the following way:

$$p_n = a[\alpha_{n-2} + \alpha_{n-1} + \alpha_n], \tag{5}$$

a is a constant.

In this work, tracking analysis of variable XE-NLMF algorithm [7] is carried out in the presence of both random and acyclic nonstationarities.

2. SYSTEM MODEL AND PERFORMANCE MEASURE

In this section, a general system model is presented which includes both types of nonstationarities, that is random and cyclic ones.

To start, consider the noisy measurement d_n that arises in a model of the form:

$$d_n = \mathbf{x}_n^T \mathbf{w}_n^o e^{j\Omega n} + \xi_n, \qquad (6)$$

where ξ_n is the measurement noise and $\mathbf{w}_n^o$ is the unknown system to be tracked. The multiplicative term $e^{j\Omega n}$ accounts for a possible frequency offset between the transmitter and receiver carriers in a digital communication scenario. Furthermore it is assumed that the unknown system vector $\mathbf{w}_n^o$ is randomly changing according to:

$$\mathbf{w}_n^o = \mathbf{w}^o + \mathbf{q}_n, \qquad (7)$$

where $\mathbf{w}^o$ is a fixed vector, and $\mathbf{q}_n$ is assumed to be a zero-mean stationary random vector process with a positive definite autocorrelation matrix $\mathbf{Q}_n = E[\mathbf{q}_n \mathbf{q}_n^T]$. Moreover, it is also assumed that the sequence $\{\mathbf{q}_n\}$ is mutually independent of the sequences $\{\mathbf{x}_n\}$ and $\{\xi_n\}$. Thus, from the generalized system model given by Equations (6) and (7), it can be seen that the effects of both cyclic and random system nonstationarities are included in this system model.
In the steady-state analysis of adaptive algorithms, an important measure of performance is their steady-state mean-square-error (MSE), which is defined as:

$$\begin{aligned} \text{MSE} &= \lim_{n\to\infty} E[e_n^2] & (8)\\ &= \lim_{n\to\infty} E\big\{[\xi_n + \mathbf{x}_n^T \mathbf{v}_n]^2\big\}, & (9)\end{aligned}$$

where $\mathbf{v}_n$ is the weight-error vector defined as:

$$\mathbf{v}_n = \mathbf{w}_n^o e^{j\Omega n} - \mathbf{w}_n. \qquad (10)$$

Also of interest, is the steady-state excess mean-square-error (EMSE), denoted by ζ and given by:

$$\zeta = \lim_{n\to\infty} E\big\{[\mathbf{x}_n^T \mathbf{v}_n]^2\big\}. \qquad (11)$$

3. FUNDAMENTAL ENERGY CONSERVATION RELATION

The fundamental energy conservation relation [4] is presented next. Using Equation (1) and Equation (7), the following recursion is obtained:

$$\mathbf{v}_{n+1} = \mathbf{v}_n - \gamma_{xe} \mathbf{x}_n^* f(e_n) + \mathbf{c}_n e^{j\Omega n}, \qquad (12)$$

where $\mathbf{c}_n$ is defined as:

$$\mathbf{c}_n = \mathbf{w}^o(e^{j\Omega} - 1) + \mathbf{q}_{n+1} e^{j\Omega} - \mathbf{q}_n. \qquad (13)$$

Now, let's define the following a-priori estimation error, $e_{an} = \mathbf{x}_n^T \mathbf{v}_n$ and a-posteriori estimation error, $e_{pn} = \mathbf{x}_n^T(\mathbf{v}_{n+1} - \mathbf{c}_n e^{j\Omega n})$. Then, it is very easy to show that the estimation error and the a priori error are related via $e_n = e_{an} + \xi_n$.

Also, the a-posteriori error is defined in terms of the a priori error as follows:

$$e_{pn} = e_{an} - \frac{\gamma_{xe}}{\hat{\mu}_n} f(e_n), \qquad (14)$$

where $\hat{\mu}_n = 1/\|\mathbf{x}_n\|^2$. Substituting Equation (14) into Equation(12) results into the following update relation:

$$\mathbf{v}_{n+1} = \mathbf{v}_n - \hat{\mu}_n \mathbf{x}_n^* [e_{an} - e_{pn}] + \mathbf{c}_n e^{j\Omega n}. \qquad (15)$$

By evaluating the energies of both sides of the above equation (taking into account that $\hat{\mu}_n \|\mathbf{x}_n\|^2 = 1$) , the following relation is obtained:

$$\| \mathbf{v}_{n+1} - \mathbf{c}_n e^{j\Omega n} \|^2 + \hat{\mu}_n |e_{an}|^2 = \| \mathbf{v}_n \|^2 + \hat{\mu}_n |e_{pn}|^2. \qquad (16)$$

It can be seen that if $\Omega = 0$ (i.e., no frequency offset between the transmitter and the receiver), the above equation reduces to the basic fundamental energy conservation relation.

4. TRACKING ANALYSIS

The energy relation (16) will be used to evaluate the excess-mean-square error at steady state. But before starting the analysis, first the following assumptions are stated:

A1 In steady-state, the weight error vector $\mathbf{v}_n$ takes the generic form $\mathbf{z}_n e^{j\Omega n}$, with the stationary random process $\mathbf{z}_n$ independent of the frequency offset Ω.

A2 The noise ξ_n is a zero-mean iid process, and is independent of the input process. This assumption is justified in several practical examples.

Using Equation (14), assumption $\mathbf{A1}$, and taking expectation of both sides of Equation (16) and the fact that at steady state $E[\mathbf{v}_{n+1}] = E[\mathbf{v}_n]$, the following relation can be obtained:

$$\begin{aligned} E[\hat{\mu}_n \parallel e_{an} \parallel^2] = \;& 2tr\{\mathbf{Q}_n\} + \parallel \mathbf{w}^o \parallel^2 |1 - e^{j\Omega}|^2 \\ & -2Re\{E\big[\mathbf{q}_n^*(\mathbf{z}_n - \gamma_{xe}\mathbf{x}_n^* f(e_n) e^{-j\Omega n})\big]\} \\ & -2Re\{(1 - e^{j\Omega})^* \mathbf{w}^{o*} \\ & \times E[\mathbf{z}_n - \gamma_{xe}\mathbf{x}_n^* f(e_n) e^{-j\Omega n}]\} \\ & + E\Big[\hat{\mu}_n |e_{an} - \frac{\gamma_{xe}}{\hat{\mu}_n} f(e_n)|^2\Big], \qquad (17)\end{aligned}$$

which can be used to solve for the steady-state excess-mean-square error (EMSE).

To find the value of $\mathbf{z} = E[\mathbf{z}_n]$, Equation (12) is used where it is multiplied by the term $e^{-j\Omega n}$ and then expectation is taken on both sides to get:

$$(1 - e^{j\Omega})\mathbf{z} = \gamma_{xe} E\Big(\mathbf{x}_n^* f(e_n) e^{-j\Omega n}\Big) + \mathbf{w}^o(1 - e^{j\Omega}). \qquad (18)$$

For the case of variable XE-NLMF algorithm, the function $f(e_n)$ can be approximated by:

$$f(e_n) \approx \frac{3e_{an}\xi_n^2 + \xi_n^3}{\delta + (1-\alpha_n)\|\mathbf{x}_n\|^2 + \alpha_n\|\mathbf{e}_n\|^2} \qquad (19)$$

which yields the value of $\mathbf{z}$ at steady-state:

$$\mathbf{z} \;=\; \left[\mathbf{I} - \frac{3\gamma_{xe}\sigma_w^2}{(1-e^{j\Omega})}\frac{\mathbf{R}}{c_1}\right]^{-1}\mathbf{w}^o, \qquad (20)$$

where

$$c_1 = [\delta + (1-E[\alpha_n])tr\{\mathbf{R}\} + E[\alpha_n]tr\{\mathbf{E}\}], \qquad (21)$$

$\mathbf{R} = E\big[\mathbf{x}_n\mathbf{x}_n{}^T\big]$, and $\mathbf{E} = E\big[\mathbf{e}_n\mathbf{e}_n{}^T\big]$ represent the autocorrelation of the input signal and error signal, respectively, and σ_w^2 is the noise variance.

Ultimately, the steady-state excess-mean-square error, ζ, for the variable XE-NLMF algorithm is obtained from Equation (17):

$$\zeta \;=\; \frac{\sigma_w^2}{(\sigma_w^2 - 3\gamma_{xe}\phi_w^4)}\left[tr\{\mathbf{Q}_n\mathbf{R}\} + \frac{\gamma_{xe}\phi_w^6}{12\sigma_w^2} + \frac{\beta c_1}{12\gamma_{xe}\sigma_w^2}\right]. \qquad (22)$$

where

$$\begin{aligned}
\beta &= \left|1-e^{j\Omega}\right|^2 Re\Big\{tr\big(\|\mathbf{w}^o\|^2(\mathbf{I}-2\mathbf{X}_1\mathbf{X}_2)\big)\Big\},\\
\mathbf{X}_1 &= \left[\mathbf{I} - \frac{3\gamma_{xe}\sigma_w^2}{c_1}\mathbf{R}\right],\\
\mathbf{X}_2 &= \left[(1-e^{j\Omega})\mathbf{I} - \frac{3\gamma_{xe}\sigma_w^2}{c_1}\mathbf{R}\right]^{-1}.
\end{aligned}$$

5. APPROXIMATE EXPRESSIONS FOR WHITE GAUSSIAN INPUT

For a white Gaussian input signal, the autocorrelation of the input signal $\mathbf{R} = \sigma_x^2\mathbf{I}$, and therefore:

$$tr\{\mathbf{R}\} \;=\; N\sigma_x^2, \qquad (23)$$

where N is the filter length. Thus, approximate expression for the NLMF algorithm is found to be:

$$\begin{aligned}
\zeta \;=\; & \frac{\sigma_w^2}{(\sigma_w^2 - 3\gamma_{xe}\phi_w^4)}\Bigg[\sigma_x^2 tr\{\mathbf{Q}_n\} + \frac{\gamma_{xe}\phi_w^6}{12\sigma_w^2}\\
& + \frac{N\Omega^2 \parallel \mathbf{w}^o \parallel^2}{12\gamma_{xe}\sigma_w^2}\Big(1 + \frac{2(c_2 - 3\gamma_{xe}\sigma_w^2\sigma_x^2)}{3\gamma_{xe}\sigma_w^2\sigma_x^2}\Big)\Bigg].
\end{aligned} \qquad (24)$$

where

$$c_2 \;=\; \delta + N(1-E[\alpha_n)]\sigma_x^2 + NE[\alpha_n)]\sigma_e^2. \qquad (25)$$

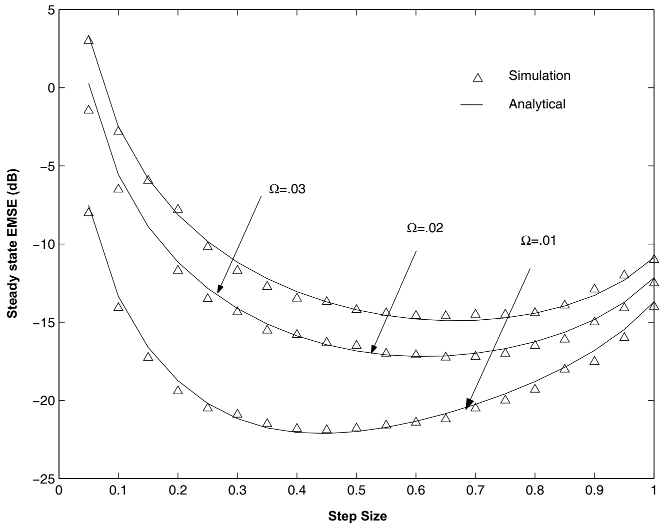

Fig. 1. Analytical and experimental ζ for Ω =0.01, Ω=0.02 and Ω=0.03, and $tr\{\mathbf{Q}_n\} = 10^{-7}$.

6. SIMULATION RESULTS

Simulation results are presented to validate the theoretical findings embodied in Equation (24), for different values of Ω. While the system characteristics are time-varying, the unknown system is given by $[1.0119 - j0.7589,\ -0.3796 + j0.5059]^T$. The signal-to-noise ratio is set equal to 10 dB and two values are considered for $tr\{\mathbf{Q}_n\}$: a very small value of $tr\{\mathbf{Q}_n\} = 10^{-7}$, and a very large one of $tr\{\mathbf{Q}_n\} = 10^{-2}$.

Figure 1 depicts the comparison between both the theoretical and simulation results for three different values of Ω, i.e., 0.01, 0.02, and 0.03. As can be seen from this Figure, close agreement between theory and simulation is obtained. This Figure shows also that the steady-state EMSE has a minimum value for a certain value of the step size γ_{xe}, e.g., for Ω =0.01, γ_{xe} is around 0.43. Also, unlike in the stationary case, the steady-state EMSE is not a monotonically increasing function of the step-size γ_{xe}. Furthermore, it is observed from this figure that degradation in performance is obtained by increasing the frequency offset Ω.

Similar behaviour is observed in Figure 2 for the case of $\Omega = 0.1$, $\Omega = 0.2$, and $\Omega = 0.3$. As expected in this case, both theory and simulation are in close agreement since here too the steady-state excess-mean-square-error of the XE-NLMF algorithm gets larger for larger values of Ω.

Figures 1 and 2 are obtained for the case when $tr\{\mathbf{Q}_n\} = 10^{-7}$ which is represents a small value. Increasing this value to 10^{-2}, the results depicted in Figure 3 for three different values of Ω, i.e., 0.1, 0.2, and 0.3, still show that the previously stated observations are similar to those obtained for a smaller value of $tr\{\mathbf{Q}_n\}$.

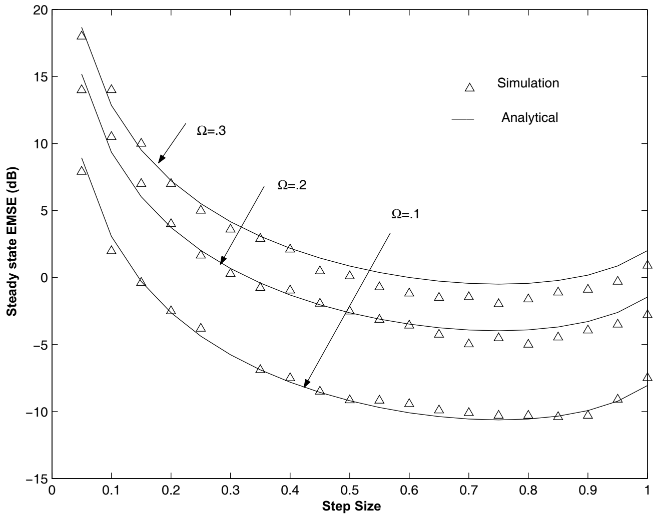

Fig. 2. Analytical and experimental ζ for Ω =0.1, Ω=0.2 and Ω=0.3, and $tr\{\mathbf{Q}_n\} = 10^{-7}$.

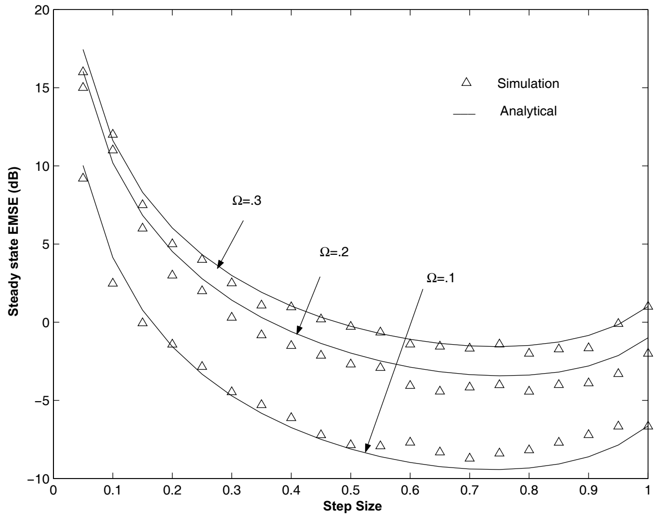

Fig. 3. Analytical and experimental ζ for Ω =0.1, Ω=0.2 and Ω=0.3, and $tr\{\mathbf{Q}_n\} = 10^{-2}$.

Finally, close agreement between theory and simulation in both cases (two different values of $tr\{\mathbf{Q}_n\}$) and different values of Ω is obtained. Also, the consistency in the performance of the steady-state excess-mean-square error of the XE-NLMF algorithm is observed on other experiments.

7. CONCLUSION

The analytical results of the steady-state EMSE are derived for the variable XE-NLMF algorithm in the presence of both random and cyclic nonstationarities. The results, show that unlike in the stationary case, the steady-state EMSE is not a monotonically increasing function of the step-size γ_{xe}, while the ability of the variable XE-NLMF algorithm to track the variations in the environment degrades by increasing the frequency offset Ω.

Acknowledgment The authors would like to acknowledge the support of King Fahd University of Petroleum & minerals.

8. REFERENCES

[1] T.S. Rappaport. *Wireless Communications*. Prentice Hall, New Jersey, USA, 1996.

[2] M. Rupp. LMS tracking behavior under periodically changing systems. *Proceedings European Signal Processing Conference, Island of Rhodes, Greece*, 1998.

[3] S. Haykin. *Adaptive Filter Theory*. Englewood Cliffs, NJ: Prentice Hall, 1996.

[4] N. R. Yousef and A. H. Sayed, “A Generalized Tracking Analysis of Adaptive Filtering Algorithms in Cyclicly and Randomly varying Environments,” *ICASSP*, pp. 440–443, 2000.

[5] N. R. Yousef and A. H. Sayed, “A unified approach to the steady-state and tracking analyses of adaptive filters,” *IEEE Transactions on Signal Processing*, Vol. 49, pp. 314–324, 2001.

[6] M. Rupp and A. H. Sayed. A Time-Domain Feedback Analysis Of Filtered-Error Adaptive Gradient Algorithms. *IEEE transactions on Signal Processing.*, 44:1428–1439, 1996.

[7] M.K. Chan, A. Zerguine, and C.F.N. Cowan. “An optimised normalised LMF algorithm for sub-Gaussian noise” *IEEE International Conference on Acoustics, Speech, and Signal Processing. Proceedings. (ICASSP ’03)* , Vol. 6, pp. VI - 377-80, 6-10 April, 2003.

[8] E. Walach and B. Widrow, “The Least Mean Fourth (LMF) Adaptive Algorithm and its Family,” *IEEE Trans. Inf. Theory, vol.* IT-30, pp. 275-283, Feb. 1984.

[9] A. Zerguine. A Novel Approach for the Convergence Analysis of the LMF Algorithm. *EUSIPCO-2004*, Vienna, Austria, September 6-10, 2004.

[10] A. Zerguine. Convergence behavior of the normalized least mean fourth algorithm. *Asilomar Conference on Signals, Systems and Computers, 2000.*, 275-278, 2000.