SNR IN WSS JITTERED SAMPLED SIGNAL

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ABSTRACT

In Analog to Digital Converters, the sampled values are not necessarily sampled in the desired time. Sampling may occur before or after the desired time because of using non ideal converters; therefore the output signal of converters contains a noise related to this inexact sampling time. In this paper, signal to noise ratio according to practical sampling of continuous signal will be estimated. For this estimation, we have assumed the continuous signal as a band limited, zero-mean real signal which is wide sense stationary (wss). Sampling rate is also considered equal or more than Nyquist sampling rate. It will be concluded that what condition should be applied in order to have any desirable ratio of signal to sampling noise.

1. INTRODUCTION

In the analysis of sampled data in control and communication systems, it is usually assumed that the sampling occurs at precisely known time instants. However, in practice Analogue to Digital Converters, ADCs are not perfect and have some specified tolerance, due to noise and other imperfections in the sampler. The difference in time between the actual sample time and the predetermined sample time is called jitter, which introduces error into the system and calculations.

Problems associated with jitter error have been previously analyzed either in A/D converters or D/A converters [1-9]. In some of the previous cases of study which aimed to estimate the signal to sampling noise ratio, we need to have knowledge about either the statistical properties of the input signal or probability density function of the jitter error, where having such information can be impossible in many cases. In addition some estimation is just regarded to the error occurs in the continuous reconstructed signal from the jittered sampled signal and not the error occurs in the discrete sampled signal, itself [10], [11].

In this paper signal to noise ratio in the sampled signal, according to inexact practical time-sampling will be estimated. For this estimation, we have only assumed the continuous signal, which is going to be sampled, as a band limited, zero-mean real signal which satisfies wide sense stationary (wss) characteristics and also the Nyquist sampling rate assumption. Also, we consider the jitter noise as a zero mean, white noise that we do not have any information about its probability density function. In fact we will estimate the signal to sampling-noise ratio by considering that our knowledge about the input signals and jitter error are limited to the above conditions and we do not have any information about others statistical properties. By applying mathematical calculations and estimating signal to sampling-noise ratio, a minimum boundary for signal to sampling-noise is obtained just as a function of jitter variance.

2. PRE CALCULATION

Here we only want to estimate the noise related to inexact sampling; therefore other noises such as quantization noise [12] will not be considered in the calculations, and the converter can be assumed as Analog to Discrete Converter.

If we consider $x_c(t)$ as a real wide sense stationary (wss) continuous signal, and $x_d[n]$ as a discrete signal which is the n-th sample of $x_c(t)$ in an ideal converter then [13]:

$$x_d[n] = x_c(nT_s) \qquad (1),$$

where T_s is sampling rate of the converter, which is equal or more than Nyquist sampling rate. But there is sampling noise related to inexact sampling of practical converter, so (1) will be changed to (2) in practical (non ideal) converters:

$$x'_d[n] = x_c(nT_s+\varepsilon_n) \qquad (2).$$

(2) means that there is about ε_n time difference between the ideal output of converter and the practical one in the n-th sampling. We call ε_n, sampling noise which is zero mean white noise and:

$$\text{Variance of } \varepsilon_n = E(\varepsilon_n^2) = \sigma_\varepsilon^2 \quad (\text{for all integer } n) \qquad (3).$$

This sampling noise makes output signal of converter to be different from desired one. In another section the necessary calculations will be done to estimate the ratio of signal to sampling noise.

3. MAIN CALCULATION

There is a relation between continuous signal and the sampled signal which is sampled with rate equal or more than Nyquist sampling rate [14]:

$$x_c(t) = \sum_{k=-\infty}^{\infty} x_d[k] \frac{Sin[\pi(t-kT_s)/T_s]}{\pi(t-kT_s)/T_s} \qquad (4),$$

where:

$$x_d[k] = x_c(kT_s) \qquad (5),$$

and T_s is sampling rate. With applying ($t= nT_s+\varepsilon_n$) in (4) we will obtain:

$$x_c(nT_s+\varepsilon_n) = \sum_{k=-\infty}^{\infty} x_d[k] \frac{Sin[\pi(nT_s+\varepsilon_n-kT_s)/T_s]}{\pi(nT_s+\varepsilon_n-kT_s)/T_s} \qquad (6).$$

We will define:

$$\alpha_n \overset{\Delta}{=} \varepsilon_n / T_s \qquad (7);$$

therefore α_n is a zero mean white noise with variance (σ_α^2) equal to $\sigma_\varepsilon^2/T_s^2$. By applying (7) and (2) in (6) we will have:

$$x'_d[n] = \sum_{k=-\infty}^{\infty} x_d[k] \frac{Sin[\pi(n+\alpha_n-k)]}{\pi(n+\alpha_n-k)} \quad (8).$$

(8) can be written in the form below:

$$x'_d[n] = x_d[n] * h[n] \quad (9),$$

where:

$$h[n] = \frac{Sin[\pi(n+\alpha_n)]}{\pi(n+\alpha_n)} = Sinc(n+\alpha_n) \quad (10).$$

Our main aim to find Signal to Noise Ratio (SNR) which is defined as below:

$$SNR = 10\log_{10}(\frac{P(S)}{P(N)}) \quad (11),$$

where:

$$P(S) = Power\ of\ Signal = Variance(Signal)$$
$$= \sigma_S^2 = E\{(x_d[n] - \overline{x_d[n]})^2\} \quad (12)$$

$$P(N) = Power\ of\ Noise = Variance(Noise)$$
$$= \sigma_N^2 = E\{(N[n] - \overline{N[n]})^2\} \quad (13)$$

$$N[n] = x'_d[n] - x_d[n]$$
$$\begin{cases} x_d[n] \rightarrow Ideal\ Signal \\ x'_d[n] \rightarrow Practical\ Signal \end{cases} \quad (14).$$

To find σ_N^2 we need mean of noise or $\overline{N[n]}$ [15]:

$$\overline{N[n]} = E\{N[n]\} = E\{(x'_d[n] - x_d[n])\} \quad (15)$$

$$\Rightarrow \overline{N[n]} = E\{x'_d[n]\} - E\{x_d[n]\} \quad (16)$$

$$\Rightarrow \overline{N[n]} = E\{x_c(nT_s + \varepsilon_n)\} - E\{x_c(nT_s)\} \quad (17)$$

$$\Rightarrow \overline{N[n]} =$$
$$E_\varepsilon\left\{E_x\{x_c(nT_s+\varepsilon_n) | \varepsilon_n\}\right\} - E\{x_c(nT_s)\} \quad (18).$$

Since $x_c(t)$ is a WSS signal, then:

$$E_x\{x_c(nT_s+\varepsilon_n)|\varepsilon_n\} =$$
$$E\{x_c(nT_s)\} = \eta_x \rightarrow \quad Constant\ for\ all\ n \quad (19);$$

therefore (18) will be changed to below:

$$\overline{N[n]} = E\{\eta_x\} - \eta_x = \eta_x - \eta_x = 0 \quad (20).$$

Now we will calculate σ_N^2 [15]:

$$\sigma_N^2 = E\{(N[n] - \overline{N[n]})^2\} =$$
$$E\{N^2[n]\} = E\{(x'_d[n] - x_d[n])^2\} \quad (21)$$

$$\Rightarrow \sigma_N^2 =$$
$$E\{x'^2_d[n]\} + E\{x_d^2[n]\} - 2E\{x'_d[n]x_d[n]\} \quad (22)$$

$$\Rightarrow \sigma_N^2 = E_\varepsilon\left\{E_x\{\left(x_c^2(nT_s+\varepsilon_n)\right)|\varepsilon_n\}\right\} +$$
$$E\{x_c^2(nT_s)\} - 2E\{x'_d[n]x_d[n]\} \quad (23).$$

Again since $x_c(t)$ is a WSS signal, we will have:

$$E_x\{\left(x_c^2(nT_s+\varepsilon_n)\right)|\varepsilon_n\} = E\{x_c^2(nT_s)\}$$
$$= R_x(0) = \sigma_x^2 \rightarrow Constant\ Value \quad (24).$$

By applying (24) in (23) we will have:

$$\sigma_N^2 = E\{\sigma_x^2\} + \sigma_x^2 - 2E\{x'_d[n]x_d[n]\} \quad (25)$$

$$\Rightarrow \sigma_N^2 = 2\sigma_x^2 - 2E\{x'_d[n]x_d[n]\} \quad (26).$$

We can make $E\{x'_d[n]x_d[n]\}$ simpler [15]:

$$E\{x'_d[n]x_d[n]\} =$$
$$E_\varepsilon\left\{E_x\{\left(x_c(nT_s+\varepsilon_n)x_c(nT_s)\right)|\varepsilon_n\}\right\} =$$
$$E_\varepsilon\{R_x(\varepsilon_n)\} \quad (27),$$

where $R_x(\varepsilon_n)$ autocorrelation of $x_c(t)$. By applying (27) in (26), we will obtain:

$$\sigma_N^2 = 2\sigma_x^2 - 2E_\varepsilon\{R_x(\varepsilon_n)\} \quad (28)$$

Since ($E^2\{x(t)x(t+\tau)\} \leq E\{x^2(t)\}E\{x^2(t+\tau)\}$) [15] or in the other hand ($|R_x(\tau)| \leq |R_x(0)| = \sigma_x^2$), we will conclude from (28) that:

$$\sigma_N^2 \leq 4\sigma_x^2. \quad (29)$$

To make (28) simpler, we need to have estimation of $E_\varepsilon\{R_x(\varepsilon_n)\}$. Since ε_n is a small value around zero, $R_x(\varepsilon_n)$ can be written as bellow [15]:

$$R_x(\varepsilon_n) \cong R_x(0) + \left.\frac{\partial R_x(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \times \varepsilon_n +$$
$$\left.\frac{\partial^2 R_x(\varepsilon)}{\partial \varepsilon^2}\right|_{\varepsilon=0} \times \frac{\varepsilon_n^2}{2} \quad (30)$$

$$E_\varepsilon\{R_x(\varepsilon_n)\} \cong R_x(0)$$
$$+ \left.\frac{\partial^2 R_x(\varepsilon)}{\partial \varepsilon^2}\right|_{\varepsilon=0} \times \frac{E\{\varepsilon_n^2\}}{2} =$$
$$\sigma_x^2 + \left.\frac{\partial^2 R_x(\varepsilon)}{\partial \varepsilon^2}\right|_{\varepsilon=0} \times \frac{\sigma_\varepsilon^2}{2} \quad (31).$$

Now we should find $\left.\frac{\partial^2 R_x(\varepsilon)}{\partial \varepsilon^2}\right|_{\varepsilon=0}$. First by using (9), we will have equations below for $R_x(\varepsilon_n)$:

$$R_x(\varepsilon_n) = E_x\{x_c(nT_s+\varepsilon_n)x_c(nT_s)\}$$
$$= E_x\{x'_d[n]x_d[n]\} = E_x\{(x_d[n]*h[n])\, x_d[n]\} \quad (32)$$

$$\Rightarrow R_x(\varepsilon_n) = E_x\{\sum_{k=-\infty}^{+\infty}\left(x_d[n]x_d[n-k]h[k]\right)\} \quad (33)$$

$$\Rightarrow R_x(\varepsilon_n) = \sum_{k=-\infty}^{+\infty}\left(R_x(kT_s)h[k]\right) \quad (34).$$

Now by considering (7) and (10):

$$R_x(\varepsilon_n) = \sum_{k=-\infty}^{+\infty}\left[R_x(kT_s)\left(Sinc(k+\frac{\varepsilon_n}{T_s})\right)\right] \quad (35)$$

$$\frac{\partial^2 R_x(\varepsilon)}{\partial\varepsilon^2} = \sum_{k=-\infty}^{+\infty}\left[R_x(kT_s)\left(\frac{\partial^2 Sinc(k+\frac{\varepsilon}{T_s})}{\partial\varepsilon^2}\right)\right] \quad (36).$$

To make (36) simpler we need to consider (37) which is about the function of Sinc:

$$\frac{\partial^2 Sinc(n+\frac{\varepsilon}{T_s})}{\partial\varepsilon^2} = \begin{cases} (\frac{1}{T_s^2})\left[\frac{-\pi\, Sin(\pi(n+\varepsilon/T_s))}{(n+\varepsilon/T_s)} - \frac{2\,Cos(\pi(n+\varepsilon/T_s))}{(n+\varepsilon/T_s)^2} + \frac{2\,Sin(\pi(n+\varepsilon/T_s))}{\pi(n+\varepsilon/T_s)^3}\right] & [for\ (n+\varepsilon/T_s)\neq 0] \\ (1/T_s^2)(-\pi^2/3) & [for\ (n+\varepsilon/T_s)=0] \end{cases} \quad (37).$$

Therefore:

$$\left.\frac{\partial^2 Sinc(n+\frac{\varepsilon}{T_s})}{\partial\varepsilon^2}\right|_{\varepsilon=\bar{\varepsilon}=0} = \begin{cases} (\frac{1}{T_s^2})\left[-\frac{2\,Cos(\pi n)}{n^2}\right] = -\frac{2(-1)^n}{T_s^2\, n^2} & [for\ n\neq 0] \\ (1/T_s^2)(-\pi^2/3) & [for\ n=0] \end{cases} \quad (38).$$

By considering (36), (38) and the fact that, (38) and $R_x(kT_s)$ are even functions (since $x_c(t)$ is real; therefore $R_x(kT_s)$ gives an even function of k.), $\left.\frac{\partial^2 R_x(\varepsilon)}{\partial\varepsilon^2}\right|_{\varepsilon=0}$ will be same as below:

$$\left.\frac{\partial^2 R_x(\varepsilon)}{\partial\varepsilon^2}\right|_{\varepsilon=0} = \sigma_x^2(1/T_s^2)(-\pi^2/3) - \sum_{k=1}^{+\infty}\left[R_x(kT_s)\left(\frac{4(-1)^k}{T_s^2\,k^2}\right)\right] \quad (39).$$

We will apply (39) in (31):

$$E_\varepsilon\{R_x(\varepsilon_n)\} \cong \sigma_x^2 - \sigma_\varepsilon^2\sigma_x^2(\pi^2/6T_s^2) - \sigma_\varepsilon^2\sum_{k=1}^{+\infty}\left[R_x(kT_s)\left(\frac{2(-1)^k}{T_s^2\,k^2}\right)\right] \quad (40).$$

By considering (7), (40) will be changed as below:

$$E_\varepsilon\{R_x(\varepsilon_n)\} \cong \sigma_x^2(1-\sigma_\alpha^2(\pi^2/6)) - \sigma_\alpha^2\sum_{k=1}^{+\infty}\left[R_x(kT_s)\left(\frac{2(-1)^k}{k^2}\right)\right] \quad (41).$$

Now we can find estimation of σ_N^2, by applying (41) in (28):

$$\sigma_N^2 \cong \sigma_x^2\ \sigma_\alpha^2(\pi^2/3) + 4\sigma_\alpha^2\sum_{k=1}^{+\infty}\left[R_x(kT_s)\left(\frac{(-1)^k}{k^2}\right)\right] \quad (42)$$

$$\sigma_N^2 \leq \sigma_x^2\ \sigma_\alpha^2(\pi^2/3) + 4\sigma_\alpha^2\sum_{k=1}^{+\infty}\left[R_x(kT_s)\,\frac{1}{k^2}\right] \quad (43).$$

Since ($E^2\{x(t)x(t+\tau)\} \leq E\{x^2(t)\}E\{x^2(t+\tau)\}$) [15] or in the other hand ($|R_x(\tau)| \leq |R_x(0)| = \sigma_x^2$); therefore we will conclude from (43) that:

$$\sigma_N^2 \leq \sigma_x^2\ \sigma_\alpha^2(\pi^2/3) + 4\sigma_\alpha^2\sum_{k=1}^{+\infty}\left[\sigma_x^2\ \frac{1}{k^2}\right] \quad (44).$$

By knowing that $\sum_{k=1}^{+\infty}\left[\frac{1}{k^2}\right] \cong 1.6449$ we can make (44) simpler as below:

$$\sigma_N^2 \leq \sigma_x^2\ \sigma_\alpha^2(\pi^2/3 + 4\times 1.6449) \cong 9.8695\sigma_x^2\ \sigma_\alpha^2 \quad (45);$$

therefore we can estimate SNR, by applying (45) in (11):

$$SNR = 10\log_{10}(\frac{P(S)}{P(N)}) = 10\log_{10}(\frac{\sigma_x^2}{\sigma_N^2}) \geq 10\log_{10}(\frac{1}{9.8695\sigma_\alpha^2}) \quad (46).$$

We also can obtain another inequality for SNR by applying (29) in (11):

$$SNR = 10\log_{10}(\frac{P(S)}{P(N)}) = 10\log_{10}(\frac{\sigma_x^2}{\sigma_N^2}) \geq 10\log_{10}(\frac{1}{4}) \quad (47).$$

Since both (46) and (47) are true, therefore we can have estimation for SNR as below:

$$SNR \geq 10\log_{10}\left(Max(\frac{1}{4}\ ,\frac{1}{9.8695\sigma_\alpha^2})\right) \quad (48)$$

or

$$SNR \geq 10\log_{10}\left(Max(\frac{1}{4}\ ,\frac{1}{10\ \sigma_\alpha^2})\right) \quad (49).$$

4. CONCLUSION

Fig.1 plots Minimum of SNR versus of σ_α^2, according to both (46) and (47). Fig.2 also plots Minimum of SNR versus of σ_α^2, but according to (49).

By concentrating on Fig.1 and Fig.2 it can be understood that when σ_α^2 is larger than 0.4, (47) dominates (46). Also for σ_α^2 smaller than 0.1, SNR is certainly greater than zero, that means the power of signal is greater than the power of noise. By considering (7), σ_α^2 is equal to $\sigma_\varepsilon^2 / T_s^2$ where σ_ε^2 is variance of sampling error; therefore if variance of sampling error (σ_ε^2) is smaller than $T_s^2/10$ then we can be sure that the power of signal is greater than the power of noise, and if variance of sampling error (σ_ε^2) is smaller than $T_s^2/100$, then the power of signal is more than 10 times greater than the power of noise.

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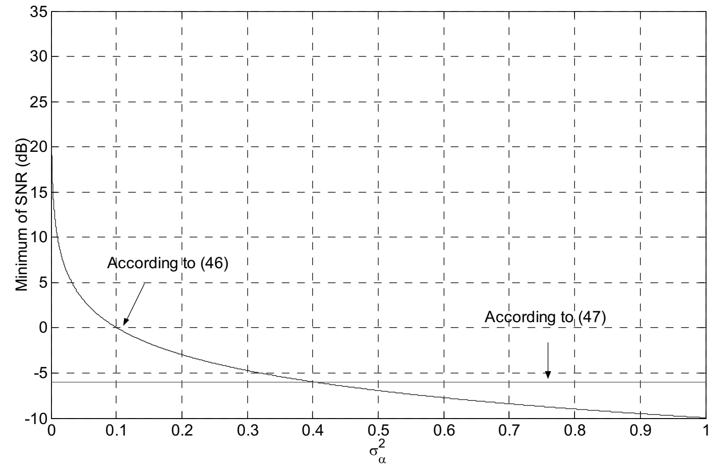

Fig.1: Minimum of SNR versus of σ_α^2 according to (46) and (47)

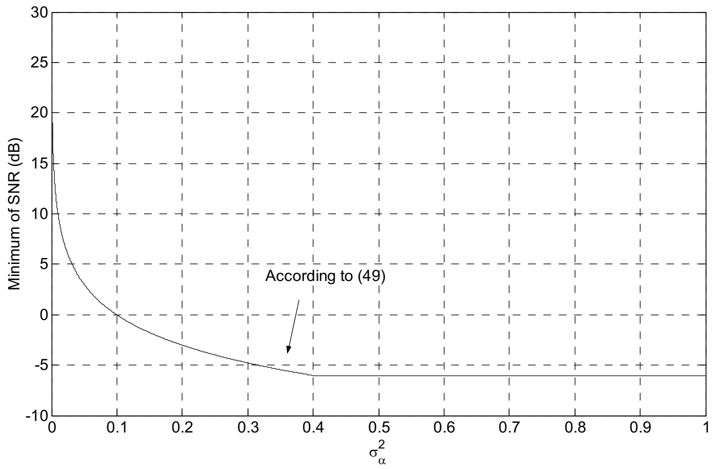

Fig.2: Minimum of SNR versus of σ_α^2, according to (49)

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