LINEAR AND QUADRATIC FUSION OF IMAGES: DETECTION OF POINT SOURCES

M. López-Caniego[1,2], *J.L. Sanz*[1], *D. Herranz*[1], *R. B. Barreiro*[1], *J. González-Nuevo*[3]

[1]Instituto de Física de Cantabria, Avda. de Los Castros s/n, 39005 Santander, Spain, e-mail: caniego@ifca.unican.es
[2] Dpto. de Física Moderna, Univ. de Cantabria, Avda. de Los Castros s/n, 39005 Santander, Spain,
[3] Dpto. de Física, Univ. de Oviedo, Avda. Calvo Sotelo s/n, 33007 Oviedo, Spain.

ABSTRACT

In this work we consider the linear and quadratic fusion of a set of n-dimensional images that contain a signal of localized compact sources embedded in a background. We aim to produce a single image that amplifies the signal and minimizes the noise. Moreover, we compare two methods to decompose the images into subimages by means of multiscale wavelet analysis. We use the Mexican hat wavelet family (MHWF), a family obtained applying iteratively the Laplacian to the standard Mexican hat wavelet (MHW). The first method uses this family as a filter (FM), operating at different scales. The second is a pyramidal method called the undecimated multiscale method (UMM). As application we consider the detection of galaxies in Cosmic Microwave Background radiation maps for the case of ESA's 44GHz Planck satellite channel using a standard linear detector. Assuming a 5   detection method, the *linear* and *quadratic* fusion techniques, together with the UMM or the FM, will improve the number of detected sources $\approx 45\%(100\%)$ as compared with the standard MHW at the optimal scale, allowing a $5\%(10\%)$ of false alarms in the total number of detections.

1. INTRODUCTION

The detection of point sources (PS) in images of the sky is one of the most challenging aspects of modern Astronomy. For example, at microwave wavelengths we observe the Cosmic Microwave Background (CMB), a fossil radiation coming from the early Universe, and, superimposed on it, several *foreground* emissions due to both Galactic and extragalactic processes. Among the extragalactic foregrounds, the most important is made up of far galaxies that, due to their small angular size (much smaller than the detector's angular resolution), appear as pointlike objects (hence the term *point sources*) convolved with the detector beam. Usually, the detector beam has a Gaussian shape, so the PS appear as small Gaussian-shaped spots embedded in the CMB and the rest of the foregrounds.

Since extragalactic PS in CMB images are very faint, their detection is often a difficult task. Several techniques have been proposed so far in the literature that are oriented to facilitate PS detection. Wavelets, for example, are well suited to the problem because of their scaling properties, that can be used to isolate and enhance features with a given characteristic scale (as is the case of PS) while keeping the information about their position. The Mexican hat wavelet has been successfully applied to CMB experiments in order to enhance the contrast between PS and the CMB and the other foregrounds [1].

Another interesting possibility appears when several different images containing the same PS are available, for example if we have images of a region of the sky taken at different wavelengths. Then it is possible to perform a fusion [2],[3] of the images into a single image where the PS are enhanced with respect to the other emissions. There are different ways to combine the images: pixel level [4], block level [5] and image decomposition [6],[7]. The combination can be linear or non-linear [8].

Image fusion is usually applied to multiple detector problems (such as the case of the sky observed at different wavelengths). Sometimes, however, it may be interesting to study the case where a single image is decomposed into several subimages and then those subimages are fused to obtain a new image where the PS are easier to detect. For instance, let us consider the case of an image decomposed with wavelets. This decomposition can be done with a multiscale method or just by filtering the image with the wavelets at a number of different scales. Then, the wavelet coefficients of each subimage will contain different information about the original image. For example, in a pyramidal multiscale decomposition each subimage will contain information about a given scale in the original image. Instead of a perfect reconstruction of the image, one can aim at reconstructing as much as possible the PS, but not the other components. Thus, a decomposition-plus-fusion scheme can be used to first identify the different elements that are present in the image and then select among them those that optimize the detection of the PS. In this work we will explore two decomposition schemes based on the Mexican hat wavelet family (MHWF) [9] followed by a subsequent fusion in order to optimize the detection of PS. Both linear and quadratic fusion will be considered.

The overview of this work is as follows: in section 2 we describe the method for combining a set of n-dimensional images using a linear and a quadratic approach. In section 3 we present a scheme to produce the subimages needed for the fusion, the MHWF used as a filter and the undecimated multiscale method. In section 4 we apply these techniques to the interesting case of detecting PS embedded in color noise for the realistic case of CMB maps. Finally, in section 5 we summarise our results.

2. LINEAR AND QUADRATIC FUSION

Let $d^i(\vec{x})$ be N images in n-dimensional space ($i = 1,\dots,N$, $\vec{x} \in {}^n$). Consider that these images are the superposition of a signal $s^i(\vec{x})$ and noise $n^i(\vec{x})$. We will assume that the signal is a set of PS characterised by their amplitude A and profile $^i(\vec{x}-\vec{x}_a)$, where $\vec{x}_a$ is the position of the source and the sources have a small contribution to the total power of the image. The background $n^i(\vec{x})$ is modeled by random fields with the following properties at any point $\vec{x}$

$$\langle n^i \rangle = 0, \quad \langle n^i n^j \rangle = C^{ij} = C^{ji}, \tag{1}$$

$$\langle n^i n^j n^k \rangle = 0, \quad \langle n^i n^j n^k n^l \rangle = C^{ij}C^{kl} + C^{ik}C^{jl} + C^{il}C^{jk}, \tag{2}$$

where $\langle \rangle$ means mean value either on the image or in the sense of realizations of the field.

Let us focus on a concrete compact source at the origin ($\vec{x_a} = \vec{0}$) being represented by

$$s^i(\vec{x}) = A \ {}^i(\vec{x}), \quad {}^i(\vec{0}) \equiv \ {}^i, \tag{3}$$

where A is the amplitude and the profile.

2.1 Linear Fusion

In this case, we only need to assume the condition given by equation (1) that involves the mean value and the correlation between the images at the same point for the noise. We define the *linear fusion* d_L of the N images as the linear superposition

$$d_L(\vec{x}) = \ _i \ a_i d^i(\vec{x}), \tag{4}$$

where a_i are constants.

Now, we are going to express the conditions to obtain a combination such that

a) $\langle d_L(\vec{0}) \rangle = A$, i. e. $d_L(\vec{0})$ is an *unbiased* estimator of the amplitude of the source,

b) The variance of d_L has a minimum, i. e. it is an *efficient* estimator.

With these conditions the problem is reduced to the minimization of $\ ^2_L$ with respect to a_i, subject to a constraint ($a_i \ ^i = 1$). Therefore, we get the best signal to noise ratio of the sources that is attainable with a linear combination of the images.

The solution for the *linear fusion* field d_L can be written in a matrix form as

$$d_L = a^t d, \quad a \equiv \frac{C^{-1}}{{}^t C^{-1}}, \tag{5}$$

where we have introduced the column vectors $a \equiv (a^i)$ and $\equiv (\ ^i)$ ($\ ^t$ is the transpose matrix) and the symmetric matrix $C \equiv (C_{ij})$, $C = C^t$.

2.2 Quadratic Fusion

In this case we need to assume the conditions given by equations (1) and (2) that involve the mean value and the correlations, up to 4th-order, between the images at the same point for the noise. We define the *quadratic fusion* d_Q of the N images as the linear plus quadratic superposition

$$d_Q(\vec{x}) = \ _i \ a_i d^i(\vec{x}) + \ _{i,j} \ b_{ij} d^i(\vec{x}) d^j(\vec{x}), \tag{6}$$

where a_i, b_{ij} are constants. The conditions to obtain a combination that optimizes the detection of the source in an analogous way as in the case of *linear fusion* are

a) $\langle d_Q(\vec{0}) \rangle = \ A + \ A^2$, i. e. $d_Q(\vec{0})$ is a quadratic estimator of the amplitude of the source (and are two free parameters),

b) The variance of d_Q has a minimum, i. e. it is an *efficient* estimator.

Therefore, the problem is reduced to the parameter minimization (with respect to a_i and b_{ij}) of $\ ^2_Q$ with two constraints ($a_i \ ^i = \ $ and $b_{ij} \ ^i \ ^j = \ $). The result for the *fusion field* d_Q can be written in a matrix form as

$$d_Q = a^t d + d^t b d = \ p^t d + \ (p^t d)^2, \quad p \equiv \frac{C^{-1}}{{}^t C^{-1}}, \tag{7}$$

and, necessarily, the quadratic term must be proportional to the square of the linear one. Thus, the *quadratic fusion* is easy to implement by performing the linear combination d_L and adding a term that is proportional to the square of d_L, $d_Q = d_L + \ d_L^2$ (we can always take $= 1$ if the linear term is present).

3. MEXICAN HAT WAVELET FAMILY (MHWF)

Let us consider wavelets in $\ ^n$, then the decomposition of a function on this basis incorporates the local and scaling behaviour of such a function. Therefore, the continuous transform involves translations and dilations

$$(\vec{x}; \vec{b}, R) \equiv \frac{1}{R^n} \ \left(\frac{|\vec{x} - \vec{b}|}{R} \right), \tag{8}$$

where is the mother wavelet, R is the dilation scale and $\vec{b}$ is the translation. We also assume that is spherically-symmetric. Then, the wavelet coefficient is defined as

$$w(\vec{b}, R) = \int d\vec{x} \, f(\vec{x}) \ (\vec{x}; \vec{b}, R), \tag{9}$$

$$w(\vec{b}, R) = \int d\vec{q} \, e^{-i\vec{q}\vec{b}} f(\vec{q}) \ (qR), \ q \equiv |\vec{q}|, \tag{10}$$

in real and Fourier space, respectively.

We are interested in the problem of point source detection in the context of astronomical images. These objects appear as points in the sky at microwave frequencies, although in the images they are convolved with the beam of the instrument used for the observation. This beam can be approximated by a Gaussian and therefore we will concentrate on Gaussian profiles for our sources.

3.1 MHWF as a filter (FM)

The MHW is defined to be proportional to the Laplacian of the Gaussian function. If we apply the Laplacian to the MHW we obtain a new wavelet, and if we further apply this operator the result is a family that we call MHWF. In two dimensions, this family can be written as

$$_m(x) = \frac{(-1)^m}{2^m m!} \ ^m \ (x), \ x \equiv |\vec{x}|, \tag{11}$$

$$_m(q) = \frac{1}{2^m m!} q^m e^{-\frac{q^2}{2}}, \ q \equiv |\vec{q}|, \tag{12}$$

in real and Fourier space, respectively. Note that is the two-dimensional Gaussian and $\ _1$ corresponds to the standard MHW. Taking into account equation (10) and the Gaussian profile of the source in Fourier space $\ (q) = \ ^2 \exp(-(q \ )^2/2)$, the expression of the wavelet coefficient for a Gaussian source filtered with the MHWF at the order m and at the scale R is

$$\frac{w_m}{T_0} = \frac{y^2}{(1+y^2)^{1+m}}, \quad y \equiv \frac{R}{\ }, \tag{13}$$

where T_0 is the amplitude of the source and is the beam width (i.e. the width of the profile).

This family allows to decompose an image into subimages simply by applying the wavelets given by equation (11) at the order m and scale R on the image. Note that R can take any real value (not only integer numbers). The filter scale R can be optimized in order to obtain the maximum enhancement of the PS with respect to the background [1]. Therefore, we are using a filtering method (FM).

3.2 MHWF & Undecimated Multiscale Method (UMM)

We can generalize the MHW on the plane and obtain three isotropic filters for which the distance is the natural scale variable to be dilated at any point [9]. The first two filters, $_{vh}$ and $_{d}$, are given by the first and second order Laplacian of the Gaussian filter, the usual MHW and the *"diagonal Mexican hat wavelet"* (DMHW). The third one, $_{c}$, is called the *"complementary Mexican hat wavelet"* (CMHW) and is such that we have a perfect reconstruction of any function in ^{2} at any scale using the Gaussian filter as scaling function. Note that the MHW and the DMHW have different normalizations compared with the wavelets from the previous subsection. In this way, we introduce a non-orthogonal, overcomplete basis. These functions are given in polar coordinates $(x, \ )$ for any fixed and arbitrary point on ^{2} by

$$\begin{aligned} (x) &= e^{-\frac{x^2}{2}}, \quad {}_{vh}(x) = \left(1-\frac{x^2}{2}\right)(x), \\ {}_{d}(x) &= \left(1-\frac{x^2}{2}+\frac{x^4}{8}\right)(x), \\ {}_{c}(x) &= (\vec{x}) - \left(3-\frac{3x^2}{2}+\frac{x^4}{8}\right)(x), \end{aligned}$$

where $(\vec{x})$ is the 2D Dirac distribution. Their Fourier transforms are

$$\begin{aligned} (q) &= e^{-\frac{q^2}{2}}, \quad {}_{vh}(q) = \frac{q^2}{2}e^{-\frac{q^2}{2}}, \\ {}_{d}(q) &= \frac{q^4}{8}e^{-\frac{q^2}{2}}, \\ {}_{c}(q) &= 1-\left(1+\frac{q^2}{2}+\frac{q^4}{8}\right)e^{-\frac{q^2}{2}}, \end{aligned} \tag{14}$$

The CMHW allows any function to be exaclty reconstructed and is defined as $-(Gaussian+MHW+DMHW)$, where is the Dirac distribution.

The UMM [10] is a pyramidal method that allows one to decompose any image $f(\vec{q})$ at any scale (multiscale analysis), using the analysing wavelets $_{vh}$, $_{d}$ and $_{c}$. In particular, for a pixelized image with pixel size l_p, we filter the image at this scale $R_1 = l_p$ and the image is decomposed, in Fourier space, as follows

$$\begin{aligned} f(\vec{q}) &= w_{vh}(R_1) + w_d(R_1) + w_c(R_1) + w_s(R_1), \\ w_{vh}(R_1) &= {}_{vh}(qR_1)f(\vec{q}), \quad w_d(R_1) = {}_{d}(qR_1)f(\vec{q}), \\ w_c(R_1) &= {}_{c}(qR_1)f(\vec{q}), \quad w_s(R_1) = (qR_1)f(\vec{q}), \end{aligned}$$

Table 1: MHWF used as a filter (FM) and 5 detector. N_{det} denotes the average number of real detections, N_f the average number of false alarms, $r \equiv 100 \times N_f/(N_{det}+N_f)$.

	0	2500	5500	9750
N_{det}	6.26	7.42	8.85	10.90
N_f	0.06	0.18	0.60	2.35
$r(\%)$	0.95	2.37	6.34	17.70

where $w_s(R_1)$ is the approximation image. Note that for simplicity we do not write explicitly the dependence of w on q. Then we apply the wavelet family at the scale $R_2 = 2l_p$ to the approximation image $w_s(R_1)$, and continue the scheme until the scale R_n. Therefore, the image $f(\vec{q})$ can be analysed and decomposed in different scales nl_p

$$f(\vec{q}) = \sum_i [w_{vh}(R_i) + w_d(R_i) + w_c(R_i)] + w_s(R_n) \tag{15}$$

The UMM has two very interesting characteristics: i) it is a multiscale approach that allows one to study different resolution levels of the image, preserving the number of pixels at any level, ii) the family of wavelets is isotropic. Hereinafter, we will assume that the pixel size is $l_p = 1$.

4. APPLICATION: DETECTION OF POINT SOURCES

The aim of this work is to improve the detection of PS. We generate 500 realistic simulations with the specifications of the 44 GHz channel of ESA's Planck satellite. The images are 128x128 pixels in size (6 arcmin/pixel) and contain the so-called Cosmic Microwave Background radiation, PS distributed spatially following a Poisson distribution and with intensity according to [11] and a number of diffuse emissions from our galaxy. These components are convolved with the response of the instrument, that can be modeled by a Gaussian beam of FWHM=24 arcmin. In addition, we add instrumental noise. Note that the average signal-to-noise is ≈ 2.3.

We compare the two methods of decomposing an image into subimages. On the one hand, the MHWF used as a filter, and on the other, the UMM. In both cases we use the linear and quadratic techniques from section 2 to fuse the subimages and then look for sources above the 5 threshold. In addition, we compare the results with those obtained filtering the original image with the standard MHW at the optimal scale (MHWopt), a standard tool for detecting PS in astronomy [1]. In Table 1 we present the average number of real detections (N_{det}), false alarms (N_f) and ratio (r) for the FM case when detecting directly at the 5 threshold on the *quadratic* combination. Note that $= 0$ corresponds to the linear case. We have used as filters the three first members of the MHWF, obtained using eq. (11) with $m = [1,2,3]$. For each filter, we determined the optimal scale R that gives the maximum enhancement of the PS: R = 1.20, 1.86 and 2.35 pixels, for m = 1, 2 and 3, respectively. We have explored different values of such that the number of false alarms can be compared with those obtained for the MHWopt for different thresholds. In Table 2 we present N_{det}, N_f and r for the UMM in an analogous way as in Table 1. In Table 3 we show N_{det}, N_f and r for the MHWopt at different levels. First, we remark that the average number of detected sources above 5

Table 2: MHWF used with the undecimated multiscale method (UMM) and 5 detector.

	0	2500	5500	9750
N_{det}	6.27	7.45	8.87	11.00
N_f	0.05	0.17	0.60	2.47
$r(\%)$	0.79	2.23	6.33	18.30

Table 3: MHW used as a filter at the optimal scale R=1.2.

	5.0	4.5	4.0	3.7	3.3
N_{det}	4.77	5.70	7.00	7.98	9.77
N_f	0.004	0.02	0.18	0.60	2.38
$r(\%)$	0.08	0.35	2.50	6.99	19.61

in the original image is $\simeq 1$, as compared with $\simeq 4.8$ obtained with the standard MHW at the optimal scale ($R = 1.2$). This shows the importance of using filters to enhance the signal and reduce the noise, and for this particular case, to remove the diffuse components efficiently.

The number of false alarms above 5 for the MHWopt case is very low. In order to compare with the *quadratic* cases, we fix N_f for different values of and lower the threshold for the MHWopt until we obtain similar numbers of N_f. Then we compare the number of real detections N_{det}. For values of of the order 2500, we detect $\simeq$ 5% more sources than with the MHWopt (4 threshold). For higher values of (4000-10000) there are $\simeq$ 10% more detections than with the MHWopt (3.3 threshold).

We remark that if we are interested in detecting a large number of real sources above 5 , and we allow a 10% of false detections, both methods yield twice as many detections as the MHWopt. Furthermore, we compare the two methods for decomposing images, FM and the UMM. We find that for the considered values of , we obtain a similar result with both methods, although the UMM seems to give a few more real detections and slightly lower ratios than the FM.

5. CONCLUSIONS

The aim of this paper is two-fold. First, we have presented a new method that combines images that contain localized sources in such a way that the output image has minimum variance and the image fusion gives at the position of the source an unbiased estimator of the amplitude. We studied the *linear* and *quadratic* fusion approach. Second, we have compared the two methods between them and the standard MHW at the optimal scale.

We have tested these ideas for realistic simulations of the 44 GHz channel of ESA's Planck satellite. We have decomposed each of the simulations using the MHWF as a filter (FM) and the undecimated multiscale method (UMM). We have then fused the linear and quadratic corresponding subimages and applied a 5 threshold to detect PS. Then we have compared the average number of true detections and false alarms for the *linear* and *quadratic* case with those obtained with the Mexican hat wavelet at the optimal scale.

When comparing the MHWopt with the *linear fusion* and *quadratic fusion*, if we fix the number of false alarms (0.6-2.5), and lower the detection threshold for the MHWopt, we see that the UMM and the FM methods detect $\sim$ 10% more real sources than the MHWopt. If we instead fix the detection threshold to 5 and we allow up to $\sim$ 10% of false alarms, the improvement for high values of is of the order 100%.

When we compare the two methods for decomposing the images, both yield similar numbers of detected sources and false alarms, although the UMM seems to give slightly better results. We remark that the parameter can be easily optimized from the simulations. In future works we plan to apply these techniques to other backgrounds, such as those found in higher frequency channels in the Planck satellite.

Acknowledgments

The authors thank F. Argüeso and L. Toffolatti for useful discussions. MLC and JGN thank the Ministerio de Educación y Ciencia (MEC) for an FPI and FPU predoctoral fellowships. RBB thanks the MEC and the Universidad de Cantabria for a RyC contract. DH acknowledges the MEC for a JdlC contract. We acknowledge partial support from the Spanish MCYT project ESP2002-04141-C03-01 and from the EU Research Training Network 'Cosmic Microwave Background in Europe for Theory and Data Analysis'.

REFERENCES

[1] P. Vielva, E. Martínez-González, L. Cayón, J. M. Diego, J. L. Sanz & L. Toffolatti, "Predicted Planck extragalactic point-source catalogue", *Monthly Notices of the Royal Astronomical Society*, vol. 326, issue 1, pp. 181-191.

[2] H. Maitre and I. Bloch, "Image fusion", *Vistas in Astronomy*, vol. 41, pp. 329-336, 1997.

[3] Y. Wang and B. Lohmann, "Multisensor image fusion: concept, method and applications", Institut für Automatisierungstechnik, Uni. Bremen, Tech. Rep., Dec. 2000.

[4] R.S. Blum, "Robust image fusion using a statistical signal processing approach", *Information Fusion*, i.p. 2004.

[5] S. Li, J.T. Kwok and Y. Wang, "Combination of images with diverse focuses using the spatial frequency", *Information Fusion* vol. 2, pp. 169-176, 2001.

[6] A. Bijaoui, Y. Bobichon and B. Vandame, "Multiscale image fusion in Astronomy", *Vistas in Astronomy*, vol. 41, pp. 365-372, 1997.

[7] G. Piella, "A general framework for multiresolution image fusion: from pixels to regions", *Information Fusion*, vol. 4, pp. 259-280, 2003.

[8] P.R. Schrater, "Bayesian data fusion and credit assignment in vision and fMRI data analysis", in *Proceedings of SPIE*, vol. 5016, pp. 24-35, 2003.

[9] J. González-Nuevo, F. Argüeso, J.L. Sanz, L. Toffolatti, P. Vielva and D. Herranz, "The Mexican hat wavelet family: application to point source detection on CMB maps", preprint.

[10] J-L. Starck, F. Murtagh and A. Bijaoui , *Image Processing and Data Analysis: the Multiscale Approach*, Cambridge University Press, U.K, 1998.

[11] L. Toffolatti, F. Argüeso Gómez, G. de Zotti, P. Mazzei, A. Franceschini, L. Danese and C. Burigana, "Extragalactic source counts and contributions to the anisotropies of the cosmic microwave background: predictions for the Planck Surveyor mission", *Monthly Notices of the Royal Astronomical Society*, vol. 297, issue 1, pp. 117-127.