BLOCK-BASED RECONSTRUCTION OF SIGNALS FROM NONUNIFORM SAMPLES

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ABSTRACT

We present a novel approach for reconstructing the uniform samples of a signal from its nonuniform samples. Two new algorithms are proposed in this context for the reconstruction of time-limited signals. These block-based methods are proved to be stable and that the reconstruction error is always bounded by the norm of the input error sources. We considered the sampling jitter and noise for this purpose. Proposed methods have good properties in terms of both theoretical and practical standpoints. The performances of the algorithms are shown through a set of simulations for a variety of signal, noise and sampling grids.

1. INTRODUCTION

Reconstruction of a signal from its nonuniform samples is an important problem of signal processing which took the attention of several researchers. An important contribution in this field is done in [1] where four theorems are presented for the reconstruction of a bandlimited signal. This work assumes that there is periodic nonuniformity in sampling and the signal is infinite length. Following research continued with this problem setting [2] and efficient multirate extensions are presented [3], [4].

In this paper, we present a new method for reconstructing the uniform samples of a signal s(t) from its nonuniform samples. In this case, we will consider a finite-length signal and we will present a block based approach. When the signal, s(t), is also bandlimited, our approach perfectly reconstructs the uniform samples of s(t). Obviously, a signal cannot be both time and bandwidth limited due to famous uncertainty principle. However, we can obtain *practically* time and bandwidth limited signals and test the reconstruction performance of different algorithms.

Proposed block-based reconstruction approach is based on the fact that there is a relation between nonuniform and uniform samples of a finite length signal which can be expressed as a linear matrix equation which is shown to be nonsingular. We use this relation and show that our method is stable in energy sense. In other words, reconstruction error is bounded and small errors or noise lead to small reconstruction errors. For this end, we prove that reconstruction error due to sampling jitter and input noise is bounded by the norm of jitter and noise errors. We present two alternatives for block-based approach and consider their performances for noisy observations. Proposed approaches are tested for non-bandlimited signals as well. It turns out that since these are block based methods, they do not have the truncation errors as in [1] when the infinite sum is approximated by a finite summation [6].

2. BLOCK BASED RECONSTRUCTION

Let us assume that $s(t)$ is an analog signal which is bandlimited to $W = \pi/T$. $s_I(\frac{T}{L}n)$ is the interpolated discrete time signal with uniform samples. $s_u(nT)$ is the uniform samples of $s(t)$ taken with Nyquist rate. $s_n(nT+\tau_n)$ is the nonuniform samples of $s(t)$ where

$$\tau_n = \lfloor p(n) \rfloor \frac{T}{L} \qquad (1)$$

and $p(n)$ is a sequence which is uniformly distributed in $[0, L)$ and $\lfloor\rfloor$ truncates the number to the closest integer. $s_n(n)$ is an aliased signal due to nonuniform sampling but it turns out that this aliasing can be completely removed and $s_u(n)$ can be obtained from $s_n(n)$.

Uniform sampling can be seen as the multiplication of the analog signal by a uniform impulse train $h(t)$ which we will denote as the sampling kernel. This operation is a convolution in frequency domain, i.e.,

$$s(t)h(t) \rightarrow S(j\Omega) * H(j\Omega) \qquad (2)$$

A similar relation exists in discrete time. But in this case, we will assume that we have finite length signals and $0 \le n \le M-1$. Therefore nonuniform samples $\bar{s}_n(n)$ are obtained as

$$\bar{s}_n(n) = s_I(n)h_n(n) \rightarrow \frac{1}{M} S_I(k) \odot H_n(k) \qquad (3)$$

where $\odot$ is the circular convolution and $S_I(k)$ and $H_n(k)$ are the M-point DFT sequences of $s_I(n)$ and $h_n(n)$ respectively. $s_n(n)$ is the nonzero samples of $\bar{s}_n(n)$. Here $h_n(n)$ is the nonuniform sampling kernel and it is given as,

$$h_n(n) = \begin{cases} 1 & \text{if } kT + \lfloor p(k) \rfloor \frac{T}{L} = n\frac{T}{L} \\ 0 & \text{otherwise} \end{cases} \qquad k = \begin{cases} \lfloor \frac{n}{L} \rfloor & \text{if } n = mL \\ 0 & \text{otherwise} \end{cases} \qquad (4)$$

The frequency domain relation in (3) can be written in matrix form as,

$$\frac{1}{M}\mathbf{H}\mathbf{W}_1\mathbf{s}_I = \mathbf{W}_1\bar{\mathbf{s}}_n \qquad (5)$$

where $\mathbf{H}$ is a circulant matrix derived from $H_n(k)$ and $\mathbf{W}_1$ is a $M \times M$ DFT matrix. Above equation can be further developed by considering two approaches. Therefore we will have two different methods for reconstruction. Method 1 depends on the interpolation by using a perfect interpolator, or the sinc function. Note that this is a symmetric and infinite length interpolation function. When we consider the delay of the convolution with this function, we can have a Toeplitz $M \times M$ $\mathbf{T}$ matrix which is generated from the sinc function. Since we have a block-based implementation, output of the convolution is also finite length. The equation in (5) can be developed for method 1 as follows,

$$\frac{1}{M}\mathbf{H}\mathbf{W}_1\mathbf{T}\mathbf{C}_1\mathbf{s}_u = \mathbf{W}_1\mathbf{C}_2\mathbf{s}_n \qquad (6)$$

$$\frac{1}{M}\mathbf{W}_1^H\mathbf{H}\mathbf{W}_1\mathbf{T}\mathbf{C}_1\mathbf{s}_u = \mathbf{C}_2\mathbf{s}_n \qquad (7)$$

$$\mathbf{B}_1\mathbf{s}_u = \mathbf{C}_2\mathbf{s}_n \qquad (8)$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are $M \times N$ ($M = LN$) interpolation kernel matrices for uniform and nonuniform interpolation respectively. Above equation can be further developed as,

$$\mathbf{B}_1^H\mathbf{B}_1\mathbf{s}_u = \mathbf{B}_1^H\mathbf{C}_2\mathbf{s}_n \qquad (9)$$

$$\mathbf{A}_1\mathbf{s}_u = \mathbf{D}_1\mathbf{s}_n \qquad (10)$$

Similar to the method 1, we can obtain the equations for the method 2. Method 2 depends on the fact that a bandlimited signal can be perfectly interpolated in DFT domain [5]. Starting with (5), we can write,

$$\frac{1}{M}\mathbf{H}\mathbf{C}_3\mathbf{W}_2\mathbf{s}_u = \mathbf{W}_1\mathbf{C}_2\mathbf{s}_n \quad (11)$$

$$\frac{1}{M}\mathbf{W}_1^H\mathbf{H}\mathbf{C}_3\mathbf{W}_2\mathbf{s}_u = \mathbf{C}_2\mathbf{s}_n \quad (12)$$

$$\mathbf{B}_2\mathbf{s}_u = \mathbf{C}_2\mathbf{s}_n \quad (13)$$

where $\mathbf{C}_3$ is the $M \times N$ interpolation matrix generated by using the approach in [5] for perfect interpolation in DFT domain and $\mathbf{W}_2$ is the $N \times N$ DFT matrix. Above equation can be put in a better form similar to the method 1 as,

$$\mathbf{B}_2^H\mathbf{B}_2\mathbf{s}_u = \mathbf{B}_2^H\mathbf{C}_2\mathbf{s}_n \quad (14)$$

$$\mathbf{A}_2\mathbf{s}_u = \mathbf{D}_2\mathbf{s}_n \quad (15)$$

The main difference between method 1 and method 2 is at the step where the interpolation of the uniform samples $s_u(n)$ is done to obtain $s_I(n)$. Given the above relations the reconstruction of the uniform samples from nonuniform samples can be done as,

$$\mathbf{s}_u = \mathbf{A}_i^{-1}\mathbf{D}_i\mathbf{s}_n \quad (16)$$

where $i = 1,2$ stands for the method used.In the following part, we will present the properties of the proposed methods for reconstruction from nonuniform samples.

3. PROPERTIES OF THE RECONSTRUCTION METHODS

Method 1 and Method 2 has certain properties which make them good candidates for nonuniform reconstruction applications. One of the important properties is the nonsingularity of the $\mathbf{A}_i$ matrices. Following Lemma and Theorems are intended to show the properties of the proposed algorithms. Proofs are skipped due to space limitation.

Lemma 1: $M \times N$ $(M \geq N)$ matrix $\mathbf{B}_1 = \frac{1}{M}\mathbf{W}_1^H\mathbf{H}\mathbf{W}_1\mathbf{T}\mathbf{C}_1$ has full column rank and $rank(\mathbf{B}_1) = N$. Also $rank(\mathbf{B}_1^H\mathbf{B}_1) = N$ and $\mathbf{A}_1 = \mathbf{B}_1^H\mathbf{B}_1$ has a well defined inverse.

Following lemma proves the nonsingularity property for the $\mathbf{A}_2$ matrix.

Lemma 2: $M \times N$ $(M \geq N)$ matrix $\mathbf{B}_2 = \frac{1}{M}\mathbf{W}_1^H\mathbf{H}\mathbf{C}_3\mathbf{W}_2$ has full column rank and $rank(\mathbf{B}_2) = N$. Also $rank(\mathbf{B}_2^H\mathbf{B}_2) = N$ and $\mathbf{A}_2 = \mathbf{B}_2^H\mathbf{B}_2$ has a well defined inverse.

Above two Lemma proves that the equations in (10) and (15) have well defined solutions. Following theorem shows that the reconstruction error due to sampling jitter and noise in nonuniform samples is bounded by the error norms.

Theorem 1: Given $\mathbf{A}\mathbf{s}_u = \mathbf{D}\mathbf{s}_n$ equation where

$$\mathbf{A} = \begin{cases} \mathbf{A}_1 & \text{if method 1} \\ \mathbf{A}_2 & \text{if method 2} \end{cases} \quad \mathbf{D} = \begin{cases} \mathbf{D}_1 & \text{if method 1} \\ \mathbf{D}_2 & \text{if method 2} \end{cases} \quad (17)$$

Assume that there is sampling jitter error indicated by $\mathbf{E}_1$ and $\mathbf{E}_2$ error matrices. Also there is sample error or noise $\mathbf{e}_n$ such that,

$$(\mathbf{A}+\mathbf{E}_1)\hat{\mathbf{s}}_u = (\mathbf{D}+\mathbf{E}_2)(\mathbf{s}_n+\mathbf{e}_n) \quad (18)$$

If $|||\mathbf{E}_1|||$ and $|||\mathbf{E}_2|||$ are sufficiently small and $(\mathbf{A}+\mathbf{E}_1)$ is full rank, then normalized reconstruction error is bounded by,

$$||\mathbf{e}|| = \frac{||\mathbf{s}_u - \hat{\mathbf{s}}_u||}{||\mathbf{s}_u||} \leq K_1|||\mathbf{E}_1||| + K_2|||\mathbf{E}_2||| + K_3||\mathbf{e}_n|| \quad (19)$$

where $|||.|||$ is a suitable matrix norm compatible with the vector norm $||.||$ and K_i are some positive constants.

Above theorem shows that the reconstruction error is bounded with respect to the sampling jitter and noise. Note that the above theorem is valid for any suitable norm and one can take advantage of this fact to prove other properties of the proposed methods. In the following theorem, we prove that the proposed reconstruction methods are stable in energy sense.

Theorem 2: (Stability) Let $s_u(n) = s(nT)$ be the uniform samples of a bandlimited signal $s(t)$ taken with Nyquist rate. Let $s_n(n) = s(nT+\tau_n)$ be the nonuniform samples of $s(t)$ and $s_n(n)+e_n(n)$ represent the noisy samples. Also $s_u(n)+e(n)$ are the noisy uniform samples after nonuniform reconstruction as given in (10) and (15). The reconstructed signal $s(t)+e(t)$ has an error $e(t)$ such that,

$$\frac{1}{T}\int_{-\infty}^{\infty}|e(t)|^2dt = \sum_{-\infty}^{\infty}|e(n)|^2 \leq \beta\sum_{-\infty}^{\infty}|e_n(n)|^2 \quad (20)$$

4. RECONSTRUCTION FROM NOISY SAMPLES

In the previous part, we have shown that the proposed methods perform well when there is noise and sampling jitter. However, it turns out that the reconstruction performance can be improved when a kind of noise filtering is used. This is a classical problem of signal processing and Wiener noise filtering methods are known to be the MSE optimum methods. In our case, there are two alternative ways for noise filtering. We can do the noise filtering before the nonuniform reconstruction or after we reconstruct the uniform samples. Below we present the Block Wiener filtering expressions for these two alternatives.

4.1 Noise filtering before reconstruction

We can implement the Block Wiener filter before the nonuniform reconstruction. In this case, we operate over the nonuniform samples,

$$\breve{\mathbf{s}}_n = \mathbf{s}_n + \mathbf{v} \quad (21)$$

Here $v(n)$ is the white noise uncorrelated with the nonuniform samples $s_n(n)$. Let $\mathbf{G}$ be the Block Wiener filter. Then,

$$\hat{\mathbf{s}}_n = \mathbf{G}\breve{\mathbf{s}}_n = \mathbf{G}\mathbf{s}_n + \mathbf{G}\mathbf{v} \quad (22)$$

Mean-square error can be expressed as,

$$E_{MSE} = \frac{1}{N}tr(E\{(\mathbf{s}_n - \mathbf{G}\breve{\mathbf{s}}_n)(\mathbf{s}_n - \mathbf{G}\breve{\mathbf{s}}_n)^H\} \quad (23)$$

If we take the derivative of this equation with respect to $\mathbf{G}^H$ and equate to zero, we obtain the block Wiener filter as,

$$\mathbf{G} = (\mathbf{R}_{s_n} + \mathbf{R}_v)^{-1}\mathbf{R}_{s_n} \quad (24)$$

4.2 Noise filtering after reconstruction

We can apply the noise removal filter after the nonuniform reconstruction. In this case, we will have slightly different equations than the previous case. Let $\tilde{\mathbf{s}}_u$ be the reconstructed signal,

$$\tilde{\mathbf{s}}_u = \mathbf{P}(\mathbf{s}_n + \mathbf{v}) \quad (25)$$

where $\mathbf{P} = \mathbf{A}^{-1}\mathbf{D}$. The output of Block Wiener filter is,

$$\hat{\mathbf{s}}_u = \mathbf{G}\tilde{\mathbf{s}}_u \quad (26)$$

and the MSE can be expressed as,

$$E_{MSE} = \frac{1}{N}tr(E\{(\mathbf{s}_u - \mathbf{G}\tilde{\mathbf{s}}_u)(\mathbf{s}_u - \mathbf{G}\tilde{\mathbf{s}}_u)^H\} \quad (27)$$

After we take the derivative of the above equation and set to zero, we obtain the block Wiener filter as,

$$\mathbf{G} = (\mathbf{P}\mathbf{R}_{s_n}\mathbf{P}^H + \mathbf{P}\mathbf{R}_v\mathbf{P}^H)^{-1}\mathbf{P}\mathbf{R}_{s_n}P^H \quad (28)$$

We have implemented both of the above approaches for noise removal. The results are presented in the following section.

5. COMPARISON OF RECONSTRUCTION METHODS

In this part, we will compare different methods with each other in terms of their MSE for reconstruction from nonuniform samples. We will consider the following methods:

a) Wiener Reconstruction: It is possible to use a Wiener filter for reconstruction from nonuniform samples. In this case, the performance certainly depends on the cross correlation between the uniform and nonuniform samples. Let the reconstruction estimate be,

$$\acute{\mathbf{s}}_u = \mathbf{G}(\mathbf{s}_n + \mathbf{v}) \tag{29}$$

Here $v(n)$ is the white noise uncorrelated with the nonuniform samples $s_n(n)$. Mean-square error can be expressed as,

$$E_{MSE} = \frac{1}{N} tr(E\{(\mathbf{s}_u - \mathbf{G}\acute{\mathbf{s}}_u)(\mathbf{s}_u - \mathbf{G}\acute{\mathbf{s}}_n)^H\} \tag{30}$$

If we take the derivative of the above expression for $\mathbf{G}^H$ and set to zero we obtain,

$$\mathbf{G} = (\mathbf{R}_{s_n} + \mathbf{R}_v)^{-1}\mathbf{R}_{s_u s_n} \tag{31}$$

It turns out that, the performance of Wiener reconstruction is limited due to the aliasing in the nonuniform samples $\mathbf{s}_n$. We should also note that we have used the estimate in (b) for finding $\mathbf{R}_{s_u s_n}$ which also degrades the performance.

b) Interpolation by lowpass filtering: It is possible to use a sharp lowpass filter to first interpolate the nonuniform samples $s_n(n)$ by L and then decimate by the same factor to obtain the estimates for $s_u(n)$. Obviously this approach also has limited success. However when the signal is highly nonbandlimited and noisy, this may result better performance in comparison.

c) Block reconstruction: This is the approach that we propose for noiseless reconstruction. We can use either Method 1 or Method 2 for this purpose. However when the nonuniform samples are noisy, performance can be improved by employing a certain type of noise filtering.

d) Noise filtering before Block reconstruction: In this case, we use a Wiener filter for noise filtering for the nonuniform samples. Then block reconstruction methods are used to obtain the uniform samples. It turns out that noise filtering before block reconstruction approach has better performance that all the other alternatives mainly due to the fact that noise amplification may be seen in certain cases if noise filtering is done after the reconstruction.

e) Noise filtering after Block reconstruction: Wiener noise removal filter is used after block reconstruction. This approach is inferior to noise filtering before reconstruction. However it has better performance than only block reconstruction (c) especially when the noise power is high or SNR is low.

5.1 Performance comparisons

We have done several experiments in order to compare the MSE performances of the proposed approaches. In these experiments, τ_n is an integer uniformly distributed in $[0, L)$. We selected $L = 10$, $N = 63$, and $M = 630$. In the first experiment, we have generated practically time and bandlimited signals by using highly selective filters. Figure 1 and 2 show the time and frequency characteristics of an example of such signals. The experiment is done with 100 trials with different signal, and noise sequences as well as nonuniform sampling instants at each case. Figure 3 shows the MSE performance of different algorithms. As it is seen, noise filtering after reconstruction has the best performance and there is not much difference between Method 1 and Method 2 for this case. Also note that we obtain almost perfect reconstruction performance as the SNR is increased.

In the second experiment, we generated time limited but nonbandlimited signals. Figure 4 and 5 show the time and frequency characteristics of such signals. We have done the experiments with different signal, noise and nonuniform sampling instants for each of the 100 trials as in experiment 1. Figure 6 shows the MSE performances of the algorithms. It turns out that method 2 outperforms method 1 in this case. Also noise filtering before block reconstruction has the best performance followed by the method 2. Since the signal is not bandlimited, MSE performances of the algorithm have a floor effect due to the aliasing. In these experiments, we have implemented method 2 for both before and after noise filtering cases.

6. CONCLUSION

We presented two alternative reconstruction methods from nonuniform samples, namely Method 1 and Method 2. These methods are block based methods and are intended for practically time and bandlimited signals. However, their performances are also good for nonbandlimited signals. In general, there are only small differences in performance between Method 1 and 2 when the input signal is practically time and bandwidth limited. When the signal is not bandlimited, Method 2 outperforms Method 1. This is mainly due to the difference between the interpolation approaches of the two methods. We have considered the noise filtering in order to improve the performance of the reconstruction methods. It turns out that noise filtering before reconstruction outperforms the noise filtering after reconstruction. In general, proposed methods have both theoretically and practically good properties and are good candidates for nonuniform reconstruction problems.

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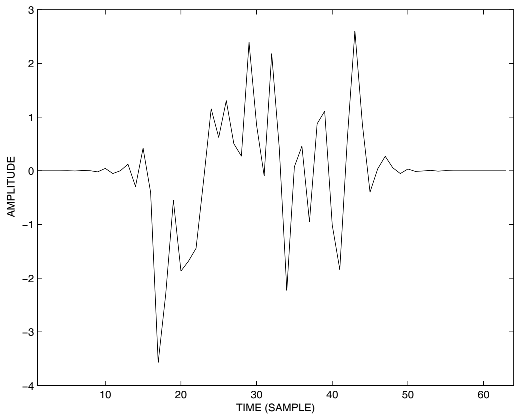

Figure 1: Uniform samples $s_u(n)$ for practically time and bandwidth limited signal.

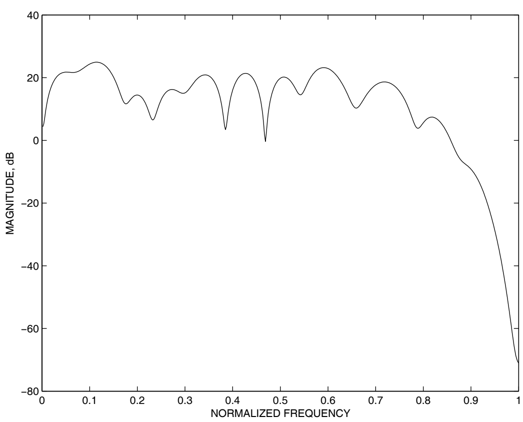

Figure 2: Frequency characteristics of $s_u(n)$.

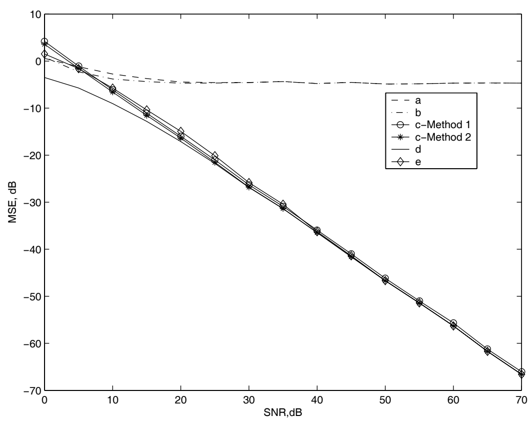

Figure 3: MSE performances of the methods in section 5.

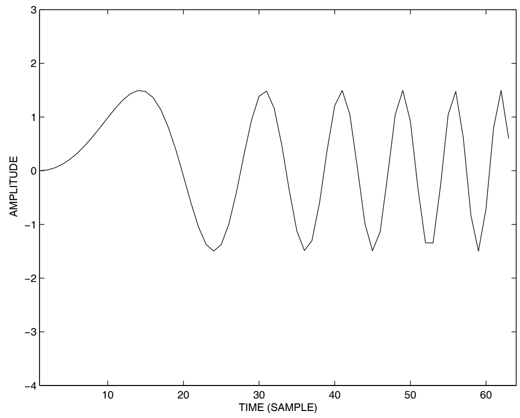

Figure 4: Uniform samples $s_u(n)$ for nonbandwidth limited signal.

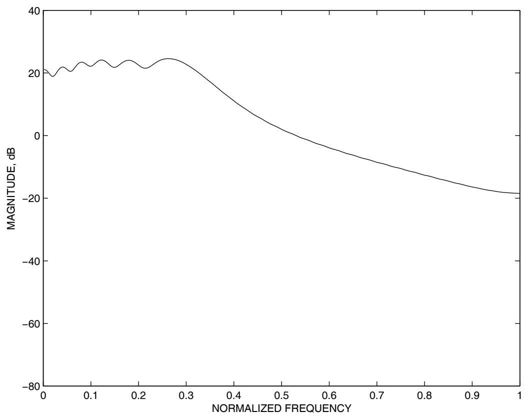

Figure 5: Frequency characteristics of $s_u(n)$.

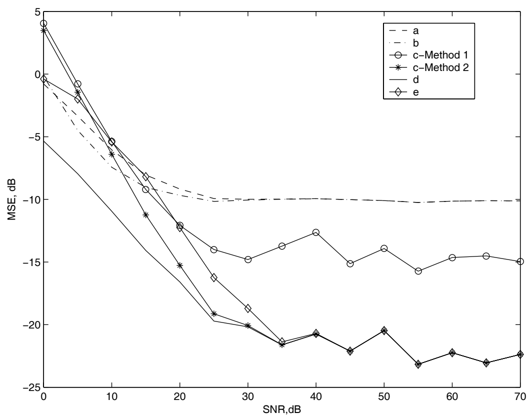

Figure 6: MSE performances of the methods in section 5 for non-bandlimited signal.