THE MEXICAN HAT WAVELET FAMILY. APPLICATION TO POINT SOURCE DETECTION IN COSMIC MICROWAVE BACKGROUND MAPS

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ABSTRACT

We propose a detection technique in 2D images based on an isotropic wavelet family. This family is naturally constructed as an extension of the Gaussian-Mexican Hat Wavelet pair and for that reason we call it the Mexican Hat Wavelet Family (MHWF). We show the performance of these wavelets for dealing with the detection of point extragalactic sources in cosmic microwave background (CMB) maps: a very important issue within the most general problem of the component separation of the microwave sky. In particular, simulations for one channel (44 GHz) of the forthcoming Planck mission have been analysed. We present the results and compare them with those obtained using the Mexican Hat Wavelet technique (MHW), which has been proven a suitable tool for detecting point sources. The MHWF provides a point source catalogue at 44 GHz of 690 sources. Under the same conditions, the MHW provides 604 sources.

1. INTRODUCTION

The component separation of the microwave sky is one of the capital problems of cosmic microwave background (CMB) data analysis. Before extracting the very important information encoded in the CMB anisotropies (e.g. cosmological parameters, evolution scenario, probability distribution of the primordial perturbations) it is critical to separate the pure CMB signal from other *contaminant* emissions also present in the microwave sky. These contaminants are known as foregrounds. Foregrounds are usually divided in two categories: galactic emissions (dust, free-free and synchrotron) and compact sources (galaxy clusters and point sources). A detailed description of the expected contamination level (both in terms of frequency and angular power spectrum) for each foreground is discussed in [1] .

The emission due to point sources (radio and infrared galaxies are seen as point-like objects due to their very small projected angular size as compared to the typical experimental resolutions) has very specific characteristics. Point sources are isotropically distributed in the sky, whereas galactic foregrounds are highly anisotropic (most of their emissions are concentrated inside the galactic plane). Moreover, galactic components have a spatial variation scale of several degrees whereas compact sources have a typical size given by the experimental beam. The detection and flux estimation of point sources at microwave frequencies is a very important task, not only for a better estimation of the CMB, but also for understanding the physics of the galaxies at these frequencies. The recent CMB data provided by the NASA satellite WMAP [2] have just started to give some information about point sources at microwave frequencies; this information should be much more important once the Planck satellite [3], which will be launched in 2007, operates.

Already pointed-out differences between point sources and other foregrounds suggest that specific techniques must be used to detect them. In particular, wavelet techniques have shown a very good performance to solve this problem. The basic idea, when we apply wavelets in R^N, is the decomposition of a function $f(\vec{x})$ on a basis that incorporates the local and scaling behavior of the function. Therefore, apart from the domain of definition, the continuous transform involves translations and dilations

$$w(\vec{b}, R) = \int d\vec{x}\, f(\vec{x}) \Psi(\vec{x}; \vec{b}, R), \qquad (1)$$

with

$$\Psi\Big(\vec{x}; \vec{b}, R\Big) \equiv \frac{1}{R^N} \psi\Big(\frac{|\vec{x} - \vec{b}|}{R}\Big), \qquad (2)$$

where ψ and w are the mother wavelet and wavelet coefficient, respectively, and R is the scale (we assume an isotropic wavelet).

A particular and relevant case is the Mexican Hat Wavelet (MHW) defined on R^2 as

$$\psi(x) \propto (1 - \frac{x^2}{2}) e^{-x^2/2}, \quad x \equiv |\vec{x}|. \qquad (3)$$

This wavelet and its extension to the sphere have been extensively used in the literature to detect structure on a 2D image: e.g. in astrophysics, for detecting point sources in CMB maps [4, 5, 6], by using the signal amplification going from real space to wavelet space.

We introduce in this paper a natural generalization of the Gaussian-MHW pair on R^2. This generalization will allow us to improve the point source detection in CMB maps. In §2, we develop our technique and we also give some properties of the Mexican Hat Wavelet Family (MHWF). In §3, we study first the case of a point source embedded in white noise, then we apply the method to the problem of point source detection in CMB maps. Realistic numerical simulations and a comparison with standard (real and wavelet space) techniques are performed. Finally, the main conclusions are drawn in §4.

2. THE MEXICAN HAT WAVELET FAMILY

The MHW on the plane is obtained by applying the Laplacian operator to the 2D Gaussian. If we apply the Laplacian to the MHW we obtain a new wavelet and if we iterate the process we get a whole family of wavelets; we call these wavelets the Mexican Hat Wavelet Family (MHWF). We choose the normalization so that the sum of the Fourier transforms of all these wavelets and the Gaussian is one. So any member of the family can be written in Fourier space as

$$\hat{\psi}_n(k) = \frac{k^{2n} e^{-\frac{k^2}{2}}}{2^n n!} \tag{4}$$

The new wavelet expression in real space is

$$\psi_n(x) = \frac{(-1)^n}{2^n n!} \triangle^n \varphi(x) \tag{5}$$

where φ is the 2D Gaussian $\varphi(x) = \dfrac{e^{-x^2/2}}{2\pi}$. Note that $\psi_1(x)$ is the standard MHW.

Since the MHW has proved very useful for dealing with point source detection, let us explore the performance of the MHWF in this practical application.

Let us consider a field $f(\vec{x})$ on the plane R^2, where $\vec{x}$ is an arbitrary point. One can define the wavelet coefficient at scale R at the point $\vec{b}$ in the form given by equations (1),(2) with N=2.
The Fourier transform of $\psi_n(x)$ is

$$\hat{\psi}_n(k) = \int_0^\infty dx\, x J_0(kx) \psi_n(x), \tag{6}$$

where $\vec{k}$ is the wave number, $k \equiv |\vec{k}|$ and J_0 is the Bessel function of the first kind.
The wavelet coefficients $w_n(\vec{b}, R)$ for each member of the MHWF can be obtained in the following form

$$w_n(\vec{b}, R) = \int d\vec{k} e^{-i\vec{k}\cdot\vec{b}} f(\vec{k}) \hat{\psi}_n(kR), \tag{7}$$

this expression can be rewritten (we assume the appropriate differential and boundary conditions for the field f) as

$$w_n(\vec{b}, R) = \int d\vec{x}\, [\triangle^n f(\vec{x})] \varphi\Big(\frac{|\vec{x} - \vec{b}|}{R}\Big). \tag{8}$$

Hence, the wavelet coefficient at point $\vec{b}$ can be interpreted as the filtering by a Gaussian window of the invariant $(2n)$ th-order differences of the field f. We are then decomposing the field(image) with this wavelet family and analyzing it at different resolution levels.

3. SOURCE DETECTION WITH THE MHWF

The Mexican Hat Wavelet (MHW) was applied with great success to the problem of detecting extragalactic sources in CMB 2D maps [4, 5]. The method was extended to the sphere [6] and applied successfully to detailed simulations of maps at the Planck frequencies and with the characteristics of the Planck experiment. In all these cases the MHW and its extension to the sphere were used as a filter at different scales, enhancing the signal to noise ratio for the sources and allowing a very efficient detection and a good determination of the source flux. In the following we will test the MHWF, as previously defined, for source detection.

A basic point when we use a filter for source detection is how the signal to noise ratio of the sources in the filtered map is increased with respect to that of the original map; this effect is called amplification. For our first example, we consider a point source with a Gaussian profile embedded in white noise (i.e. a homogeneous and isotropic random field with a constant power spectrum). We consider a pixelized image consisting of a filtered source plus white noise added at any pixel and then we calculate the amplification produced by the MHWF in this particular case. Let us first calculate analytically, see eq. (7), the wavelet coefficients obtained by filtering with any member of the MHWF a source of temperature $T(x) = T_0 e^{-\frac{x^2}{2\gamma^2}}$, where the source of temperature T_0 has been previously convolved with a Gaussian beam of dispersion γ. We obtain the following coefficients at $\vec{b} = 0$, (the point of maximum temperature of the source)

$$w_n = \frac{T_0 \beta^{2n}}{(1+\beta^2)^{n+1}} \tag{9}$$

where $\beta = \frac{R}{\gamma}$ and R is the wavelet scale. We want to point out that this formula is true for a point source embedded in any background and will allow us to estimate the source temperature or equivalently its flux in CMB maps. Now we will calculate the rms deviation σ_{w_n} of the image filtered with a wavelet of scale R as a function of the rms deviation σ of the original image

$$\sigma_{w_n} = \frac{\sigma l_p \sqrt{(2n)!}}{R \pi 2^n n!} \tag{10}$$

where l_p is the pixel size. The amplification λ_n is defined as

$$\lambda_n = \frac{w_n / \sigma_{w_n}}{T_0/\sigma} \tag{11}$$

Then, the amplification for the white noise case can be written as

$$\lambda_n = \frac{\beta^{2n+1} \gamma \pi 2^n n!}{(1+\beta^2)^{n+1} l_p \sqrt{(2n)!}} \tag{12}$$

We can obtain the scale of maximal amplification for each wavelet, just by deriving with respect to β and equating to zero. We obtain $\beta_{max} = \sqrt{2n+1}$, so that the optimal scale depends on n and is $R_{max} =$

$\sqrt{2n+1}\gamma$. The maximal amplification only depends on n and can be written

$$\lambda_{max} = \frac{(2n+1)^{(n+\frac{1}{2})} n! \pi \gamma}{2(n+1)^{n+1}\sqrt{(2n)!}\, l_p} \tag{13}$$

If we calculate λ_{max} according to this formula, we obtain a maximum value for n=0, a Gaussian filter with $R = \gamma$. So, the optimal amplification is reached when we filter again with a Gaussian beam of the same dispersion as the original one. The maximal amplification for other members of the MHWF decreases slowly as the index n increases.

We have carried out a simple calculation for the case of a point source embedded in a white noise background. However, we are more interested in the real conditions of a CMB experiment in which the source is embedded in the cosmic microwave background, the Galactic foregrounds and the detector Gaussian noise. The only way to analize the amplification given by (11) is to carry out simulations of the CMB maps with all the components and measure the rms deviations σ_{w_n} and σ. The expression for w_n is the one given by (9). We consider in this paper the specific conditions of one channel of the Planck satellite. The Planck satellite of the European Space Agency (ESA) is due to be launched in 2007 and will study with unprecedented accuracy the CMB, obtaining information of great interest about the origin and evolution of the Universe.

For detecting point sources and in order to calculate the amplification factor for a source embedded in a realistic CMB map, we have carried out simulations of CMB 2D maps of 12.8×12.8 square degrees, generated with the cosmological parameters of the standard model . We have added the relevant Galactic foregrounds (free-free, synchrotron and dust emission) by simulating them with the same assumptions as in [5]. Extragalactic point sources have been simulated by using the model presented in [7, 8] as a template. The predictions of this model are in very close agreement with current data on source number counts coming from different experiments.

Since we are specially interested in applying our new method to the maps which will be provided by the *Planck* mission in the next future, we have considered the specific conditions of one *Planck* channel: the 44 GHz channel of the LFI (Low Frequency Instrument). We have adopted a pixel size of $6'$, we have filtered the maps with a Gaussian beam of FWHM $= 24'$ and we have added a Gaussian uniform noise with a rms deviation $\sigma = 2.7 \times 10^{-6}$ in a square whose side is the FWHM extent of the beam. We have used the estimated instrument performance goals (for more details, see $http : \backslash\backslash sci.esa.int\backslash planck$).

According to formula (9) we can calculate the temperature of the point source at its central pixel in the filtered maps for each wavelet and any scale, given the original temperature. We find full agreement between the values obtained from the formulas and the results of our simulations for all the wavelets and scales involved. Our next step is to calculate the amplification for different members of the MHWF and for different values of the wavelet scale R; our goal is to find the optimal scales for different wavelets and compare them for source detection. In Figure 1 we show the amplification obtained at different scales with the four first members of the MHWF. The maximal amplification for the MHW is 2.26 and is obtained for $R = 7.2'$, for the following members MHW2, MHW3 and MHW4 they are 2.48, 2.46, 2.39 and are obtained for $R = 11', 13.6'$ and $16'$ respectively. The tests done with higher n show less amplification, this is also the case when a Gaussian filter is used. So, the best amplifications are obtained with the MHW2 and MHW3 wavelets.

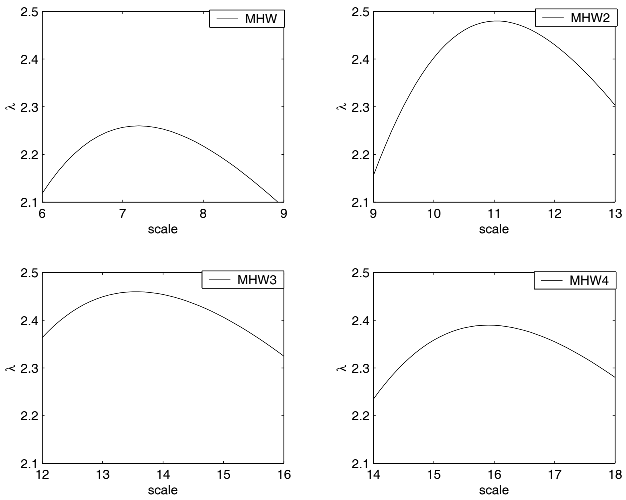

Figure 1: Amplification, λ, produced at different scales (in arcmin) and for the following wavelets: MHW (top left panel), MHW2 (top right), MHW3 (bottom left) and MHW4 (bottom right).

We have then used these two wavelets at the optimal scales to detect point sources in the maps, comparing the results with those obtained with the MHW. Our detection method is quite simple, we carry out enough simulations (126) of 2D sky patches to cover half the sky, since the Galactic plane prevents us from detecting any sources in a substantial part of the sky. As previously stated, the simulations include point sources, Galactic foregrounds, CMB and the detector noise. In each simulation we have taken into account only those pixels above 5σ in the maps obtained after filtering with the corresponding wavelets at the optimal scales. Then, we estimate the flux of the sources, according to equation (9), and compare the selected candidates with the sources simulated originally. In this first case, (5σ), there are no false detections and the number of detected sources is MHW (406), MHW2 (463) and MHW3 (458). The increase in the number of detections when we use the MHWF at the 5σ level is about 15%. If we had used the 5σ criterion in the original map, we would have detected just 119 sources. In Table 1 these results are shown together with information about the relative error in the flux estimation, the average amplification, and the flux over which the source catalogue is 95% complete.

If we consider a 4σ threshold in the wavelet filtered maps, the number of detections increase but we have a few false detections. In this second case, we have

Original Map		Filtered Map					
	$\mathbf{N}_O$			$\mathbf{N}_F$	**S(Jy)(95%)**	$\langle\delta\rangle$	$\langle\lambda\rangle$
$\mathbf{4\sigma}$	187	$\mathbf{4\sigma}$	**MHW1**	604	0.67	0.14	2.26
			MHW2	690	0.63	0.13	2.48
			MHW3	686	0.63	0.12	2.46
$\mathbf{5\sigma}$	119	$\mathbf{5\sigma}$	**MHW1**	406	0.80	0.06	2.26
			MHW2	463	0.76	0.06	2.48
			MHW3	458	0.82	0.06	2.46

Table 1: Point source detections at 44 GHz in half sky, 2π sr, for the original and filtered maps. In the first column we show the number of detected sources, N_O, at the 4σ and 5σ levels in the original map. The second column shows the total number, N_F, of detections after filtering with the different wavelets discussed in the text and by using the same detection thresholds as before. In the third column we give the fluxes at which the recovered source catalogues are complete at the 95% level. In the fourth column appear the average relative errors in the flux determination for all the detected sources. Finally, in the seventh column we present the average amplification for the detected sources.

604 sources detected with the MHW (plus 2 false detections), 690 detected with the MHW2 (plus 6 false ones) and 686 with the MHW3 (plus 7 false ones). The increase in the number of detections with respect to the MHW is also about 15%. We have proved that under the specific conditions of the 44 GHz Planck channel, the use of other members of the MHWF gives a better performance than the use of the MHW.

We want to point out that this method could be applied to the real maps provided by the Planck satellite, since we could calculate the optimal scales and the amplification in the same way with the real maps as we have done with the simulations. We can be quite confident with our simulations, since they are based in models of Galactic and extragalactic foregrounds which fit very well the data obtained by the WMAP satellite.

4. CONCLUSIONS

We have considered a natural generalization of the MHW on the plane, R^2, based on the iterative aplication of the Laplacian operator to the MHW . We have called this group of wavelets the Mexican Hat Wavelet Family (MHWF).

The MHWF can therefore be applied to the analysis of images, such as CMB maps. Our main goal is to test this wavelet family in one typical application, point source detection in CMB maps and to compare our results with those obtained by other standard techniques in wavelet and real space.

The main reason why wavelets perform well at point source detection is that they amplify the ratio between the point source intensity and the dispersion of the background. For a simple background, such as white noise, this amplification can be calculated easily, see eqs.(12) and (13). In this case the maximal amplification is obtained with the Gaussian filter.

When we consider CMB maps including point sources, CMB, Galactic foregrounds and the detector noise, such as those that will be provided by the 44 GHz Planck channel, the second and third members of the wavelet family, MHW2 and MHW3, give the maximal amplification, improving the results obtained with the MHW. The number of sources detected with these wavelets is about 15% higher than the number of detections with the MHW, these results are shown in Table 1. Our results show that it is worthwhile to use the MHWF for point source detection in CMB maps. Our method can be applied in a straightforward way to the real maps which will be provided by the Planck satellite in a few years.

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