WEIGHTED AVERAGE INSTANTANEOUS FREQUENCY BASED ON ADAPTIVE SIGNAL DECOMPOSITION

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ABSTRACT

It is often claimed that instantaneous frequency, taken as the derivative of the phase of the signal, is appropriate or meaningful only for mono-component signals, and that for multi-component signals a weighted average of individual instantaneous frequencies should be used. In this paper, we show if a signal is decomposed adaptively and we compute the matching pursuit distribution, then the first conditional spectral moment is exactly the weighted average instantaneous frequency. Two different signals will be analyzed and the above result will be illustrated in practice.

1. INTRODUCTION

Many signals exhibit time-varying frequencies, and this gives rise to the concept of instantaneous frequency (IF). It is commonly defined as the derivative ($\phi'(t)$) of the phase of the signal $z(t)=\sum_{n=0}^{+\infty} a_n(t)e^{j\phi_n(t)} = a(t)e^{j\phi(t)}$, where z(t) is in the form of the analytic signal [1]. IF is interpreted in the time-frequency literature as the average frequency at each time instant of the signal [2]. This interpretation arises because an unlimited number of time-frequency distributions ($P_z(t,\omega)$) of the signal z(t) yield the derivative of the phase for the first conditional spectral moment (i.e. $<\omega>_t = \phi'(t)$), where:

$$<\omega>_t = \frac{1}{P_z(t)} \int_{-\infty}^{+\infty} \omega P_z(t,\omega) d\omega \tag{1}$$

$$P_z(t) = \int_{-\infty}^{+\infty} P_z(t,\omega) d\omega. \tag{2}$$

Note that the well-known Wigner Ville distribution (WVD) has the above property. Although the IF, defined as the derivative of the phase of the signal, is interpreted as the average frequency at each time, it has been shown that this interpretation often does not make sense [3]. So in general, the IF of a signal, and the average frequency at each time of the signal are different quantities. A more useful quantity for the analysis of multi-component signals is the weighted average instantaneous frequency (WAIF) of the individual components [4]. For a signal $z(t)$ as defined as above, then the WAIF is obtained as follows:

$$\overline{\omega}(t) = \frac{\sum_{n=0}^{+\infty} a_n^2(t)\phi_n'(t)}{\sum_{n=0}^{+\infty} a_n^2(t)}. \tag{3}$$

This quantity has a linear relationship to the individual instantaneous frequencies and so contains the appropriate information in a more convenient way.

In this paper, we will review the matching pursuit (MP) signal decomposition and the associated bilinear distribution. Then we will derive the first conditional spectral moment, $<\omega>_t$, according to the MP distribution, and show that it is nothing more than the WAIF of the decomposed signal. Finally, two different signals will be analyzed and the above result will be illustrated in practice.

2. REVIEW OF ADAPTIVE SIGNAL DECOMPOSITION THEORY

Mallat and Zhang [5] proposed an adaptive signal decomposition. This method is based on a dictionary that contains a family of functions called elementary functions or *time-frequency atoms*. The decomposition of a signal is performed by projecting the signal over the function dictionary and then selecting the atoms which can best match the local structure of the signal. So, we compute a linear expansion of $z(t)$ over a set of elementary functions selected from the dictionary in order to best match its inner structures. The MP decomposition after M iterations can be written as follows:

$$z(t) = \sum_{n=0}^{M-1} c_n g_{\gamma_n}(t) + R^M z(t) \tag{4}$$

where $z(t)$ is the decomposed signal, and $R^M z(t)$ is the residue after M times signal decomposition. By letting $R^0 z(t) = z(t)$, then the MP algorithm decomposes the residue at each stage. Thus the original signal is projected onto a sum of elementary functions, which are chosen to best match its residues; $g_{\gamma_n}(t)$ is the *time-frequency atom* that belongs to the dictionary, and which satisfies the unitnorm requirement – i.e. $\int_{-\infty}^{+\infty}\left|g_{\gamma_n}(t)\right|^2 dt = 1$; the coefficient $c_n = < R^n z(t), g_{\gamma_n}(t) > = \int_{-\infty}^{+\infty} R^n z(t)\cdot g_{\gamma_n}^*(t)dt$ is the inner product of the functions $\left(R^n z(t), g_{\gamma_n}(t)\right)$, where "*" represents complex conjugate; and the parameter γ_n refers to the atom's parameter set. When the number of iterations is infinitive, then the residue will be zero, and so we can say

$$\lim_{M\to\infty} R^M z(t) = 0 \Rightarrow z(t) = \sum_{n=0}^{+\infty} c_n g_{\gamma_n}(t). \quad (5)$$

Mallat and Zhang [5] defined the MP distribution as follows:

$$E_z(t,\omega) = \sum_{n=0}^{+\infty}\left|c_n\right|^2 W_{g_{\gamma_n}}(t,\omega) \quad (6)$$

where c_n is complex and $W_{g_{\gamma_n}}(t,\omega)$ is the WVD of the appropriate atoms with

$$W_{g_{\gamma_n}}(t,\omega) = \int_{-\infty}^{+\infty} g_{\gamma_n}\left(t+\frac{\tau}{2}\right) g_{\gamma_n}^*\left(t-\frac{\tau}{2}\right) e^{-j\omega\tau} d\tau. \quad (7)$$

This new distribution, $E_z(t,\omega)$, can now be interpreted as an energy density function of z in the time-frequency plane. There are two ways of stopping the iterative process: one is to use a pre-specified limiting number (M) of time-frequency atoms, and the other is to check the energy of the residue, $R^M z(t)$. In this paper we will adopt the first approach to stop the adaptive signal decomposition algorithm.

3. DIFFERENT DICTIONARIES

The first conditional spectral moment of many bilinear distributions does equal the IF (derivative of phase), though it often does not make sense when we are analyzing multi-component signals. On the other hand, it was shown that IF equals the WAIF just for certain special signals [6]. We notice for the MP distribution, that what we compute as the first conditional spectral moment is identical to calculating the WAIF of the decomposed signal. Now suppose a signal is decomposed by MP, then it can be written as:

$$z(t) = \sum_{n=0}^{+\infty} c_n g_{\gamma_n}(t) = \sum_{n=0}^{+\infty} a_n(t) e^{j\phi_n(t)} = \sum_{n=0}^{+\infty} z_n(t) \quad (8)$$

where $z_n(t) = a_n(t)e^{j\phi_n(t)}$, and the parameters a_n, and ϕ_n are evaluated according to the type of chosen time-frequency elementary function. So the MP distribution can be written as follows:

$$E_z(t,\omega) = \sum_{n=0}^{+\infty}\left|c_n\right|^2 W_{g_{\gamma_n}}(t,\omega) = \sum_{n=0}^{+\infty} W_{z_n}(t,\omega). \quad (9)$$

According to (1) and (2) the first conditional spectral moment is:

$$<\omega>_t = \frac{\int_{-\infty}^{+\infty}\omega E_z(t,\omega)d\omega}{\int_{-\infty}^{+\infty} E_z(t,\omega)d\omega} = \frac{\sum_{n=0}^{+\infty}\int_{-\infty}^{+\infty}\omega W_{z_n}(t,\omega)d\omega}{\sum_{n=0}^{+\infty}\int_{-\infty}^{+\infty} W_{z_n}(t,\omega)d\omega}. \quad (10)$$

In addition, from the WVD properties [7], $\int_{-\infty}^{+\infty}\omega W_{z_n}(t,\omega)d\omega = a_n^2(t)\phi_n'(t)$ and $\int_{-\infty}^{+\infty} W_{z_n}(t,\omega)d\omega = a_n^2(t)$. Hence the first conditional spectral moment of the MP distribution takes the following form:

$$<\omega>_t = \frac{\sum_{n=0}^{+\infty} a_n^2(t)\phi_n'(t)}{\sum_{n=0}^{+\infty} a_n^2(t)}. \quad (11)$$

Obviously, this result above is identical to the WAIF (in (3)) for the decomposed signal, i.e. $<\omega>_t = \overline{\omega}(t)$. However, the convergence property of MP signal decomposition is not dependent upon the type of time-frequency atom used [8]. So the above result is valid in general, though different dictionaries will decompose a signal with different components. In the following work we will derive the first conditional spectral moment analytically, for the MP distribution based on the Gaussian as well as damped sinusoid elementary functions, and show that what is captured is indeed the WAIF.

3.1 Gaussian Dictionary

It is well known that the Gaussian atom is unique in the sense that it has the greatest "concentration" in both the time and frequency domains [9]. Such an atom is:

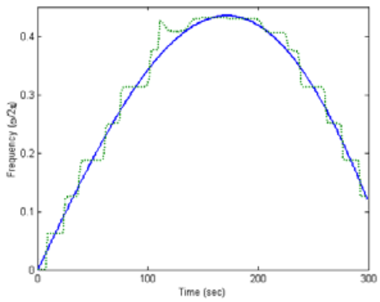
$$g_{\gamma_n}(t) = \frac{1}{\sqrt{s_n}} g\left(\frac{t-u_n}{s_n}\right) e^{j\zeta_n t} \quad (12)$$

where $g(t) = 2^{1/4}e^{-\pi t^2}$ and $\gamma_n = (s_n, u_n, \zeta_n)$ is the atom parameters set. The parameter s_n controls the envelope width of g_{γ_n}. The parameters u_n and ζ_n are the temporal placement and the frequency variable. The parameters set are all real. In addition, s_n is also positive. As we use the Gaussian dictionary for the MP signal decomposition, the index n refers to the different atoms that exist in the dictionary. Now, suppose that a signal $z(t)$ is decomposed adaptively by employing the Gaussian elementary functions as follows:

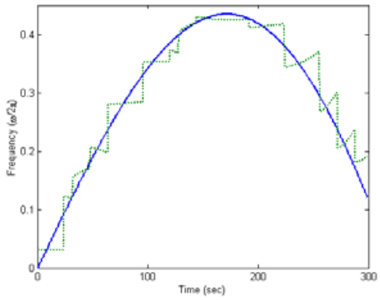
$$z(t) = \sum_{n=0}^{+\infty} c_n g_{\gamma_n}(t) = \sum_{n=0}^{+\infty} a_n(t) e^{j\zeta_n t} \quad (13)$$

where $a_n(t) = \frac{\left|c_n\right|}{\sqrt[4]{2s_n^2}}\widetilde{H}\left(\frac{t-u_n}{\sqrt{2}\,s_n}\right)$, and $\widetilde{H}(t) = e^{-2\pi t^2}$. It can be shown that the MP distribution with the Gaussian atom is:

$$E_z(t,\omega) = 2\sum_{n=0}^{+\infty}\left|c_n\right|^2 \widetilde{H}\left(\frac{t-u_n}{s_n}\right)\widetilde{F}\left(s_n(\omega-\zeta_n)\right) \quad (14)$$



(a)



(b)

Fig. 1. The true IF (solid curve) and the estimated IF (dotted curve) via the estimated first conditional spectral moment based on MP signal decomposition for: (a) the Gaussian dictionary, and (b) the damped sinusoid dictionary. The number of iterations was 16.

with $\widetilde{F}(\omega) = e^{-\frac{1}{2\pi}\omega^2}$. So from (1), we can derive the first conditional spectral moment of the bilinear matching pursuit distribution as follows:

$$< \omega >_t = \frac{1}{E_z(t)} \sum_{n=0}^{+\infty} \widetilde{c}_n \widetilde{H}\left(\frac{t-u_n}{s_n}\right) \zeta_n \qquad (15)$$

where $E_z(t) = \sum_{n=0}^{+\infty} \widetilde{c}_n \widetilde{H}\left(\frac{t-u_n}{s_n}\right)$, and $\widetilde{c}_n = \frac{|c_n|^2}{s_n}$. The WAIF can also be obtained using (3), but obviously it gives the same expression as (15).

3.2 Damped Sinusoids Dictionary

In most applications $g(t)$ is typically an even-symmetric window, such as the Gaussian window described above. An alternative is the damped sinusoidal atom [10]:

$$g_{\gamma_n}(t) = k_n b_n^{t-u_n} U(t-u_n) e^{j\xi_n(t-u_n)}, \quad 0 < b_n < 1 \qquad (16)$$

where $\gamma_n = (b_n, u_n, \zeta_n)$ is the parameters set, and $k_n = \sqrt{-2\ln(b_n)}$. The parameter $0 < b_n < 1$ is used to control the envelope width of g_{γ_n}; the parameter u_n refers to the temporal placement; ζ_n is the frequency variable; and $U(t)$ is the unit step function:

$$U(t) = \begin{cases} 1; t \geq 0 \\ 0; t < 0. \end{cases} \qquad (17)$$

Now consider that we have decomposed a signal adaptively based on using the damped sinusoid elementary function. The decomposition can now be defined as:

$$z(t) = \sum_{n=0}^{+\infty} c_n g_{\gamma_n}(t) = \sum_{n=0}^{+\infty} a_n(t) e^{j\zeta_n(t-u_n)} \qquad (18)$$

where $a_n(t) = k_n |c_n| \overline{H}_n(t-u_n)$ and $\overline{H}_n(t) = b_n^t\,\mathrm{U}(t)$. We have computed the first conditional spectral moment of the new bilinear distribution as:

$$< \omega >_t = \frac{1}{E_z(t)} \sum_{n=0}^{+\infty} \overline{c}_n \overline{H}_n^2(t-u_n) \zeta_n \qquad (19)$$

where $E_z(t) = \sum_{n=0}^{+\infty} \overline{c}_n \overline{H}_n^2(t-u_n)$, and $\overline{c}_n = k_n^2 |c_n|^2$. It is easy to show that the first conditional spectral moment, $< \omega >_t$, is the same as the WAIF as defined in (3), with $\phi_n'(t) = \zeta_n$ and $a_n(t)$ is as above.

We will now decompose both a mono-component, and a multi-component signal in practice. We will then compute the first conditional spectral moments according to (15) and (19) and examine how far they are from both the true IF and true WAIF.

Example. 1

For the first example, we consider a signal that can be categorized as mono-component, with unit amplitude and oscillatory phase:

$$z(t) = \exp\left(j0.625\pi t \sin\left(0.002\pi t\right)\right), t = 0,1,\cdots,299\,. \quad (20)$$

We decomposed the signal and set the number of algorithm iterations to be equal to 16. So, we have found the best Gaussian and damped sinusoid atoms individually. The first conditional spectral moments have been computed and are shown in Fig. 1. As we know the mathematical function for the signal, we have also obtained the true IF (phase derivative, $\phi'(t)$) and sketched this in each figure. We must remember that for a mono-component signal such as this, then the WAIF is same as the IF.

Example 2

If we now consider a two-component signal $z(t) = a_1(t)e^{j\phi_1(t)} + a_2(t)e^{j\phi_2(t)} = a(t)e^{j\phi(t)}$, then the IF is [4]:

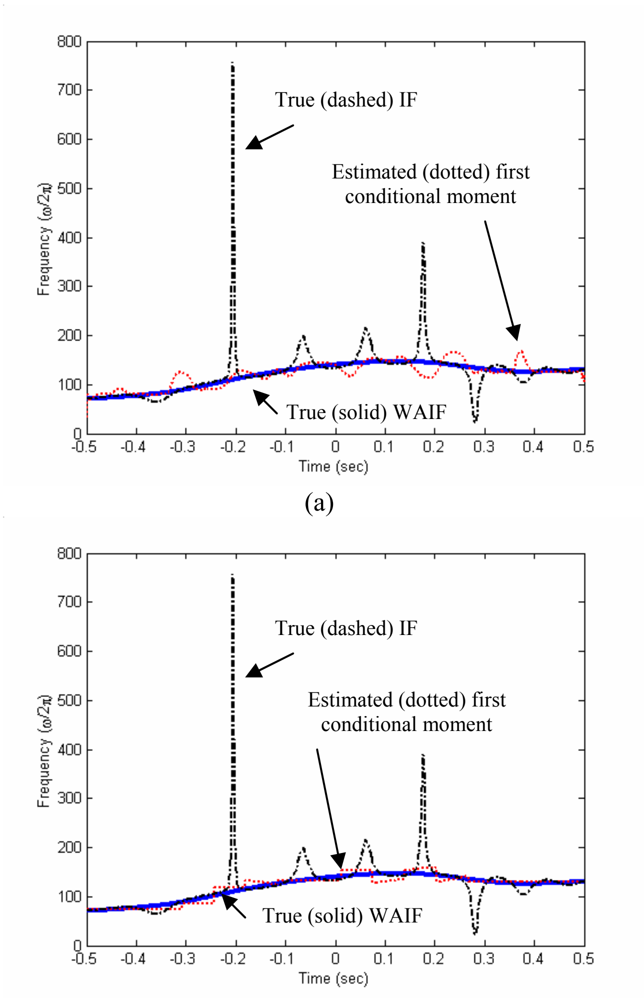
$$\begin{aligned} \phi'(t) = \frac{1}{a^2(t)} \times \Big[& a_1^2(t)\phi_1'(t) + a_2^2(t)\phi_2'(t) \\ & + a_1(t)a_2(t)\left(\phi_1'(t) + \phi_2'(t)\right)\cos(\Delta\phi_{12}(t)) \\ & + \left(a_1'(t)a_2(t) - a_2'(t)a_1(t)\right)\sin(\Delta\phi_{12}(t)) \Big] \end{aligned} \qquad (21)$$

where $a^2(t) = a_1^2(t) + a_2^2(t) + 2a_1(t)a_2(t)\cos(\Delta\phi_{12}(t))$, and $\Delta\phi_{12}(t) = \phi_1(t) - \phi_2(t)$. This IF is an oscillating function, which is very sensitive to the amplitudes of the components. So consider the following two-component signal:

$$z(t) = 0.5e^{j2\pi(100t+30t^2)} + e^{-15t^2} e^{j2\pi(150t+50t^2)}. \qquad (22)$$

It includes two chirps of unequal chirp rate and unequal time-varying amplitude (the same signal was used in [4]).

The time range is considered to be: $-0.5 \le t \le 0.5$. We have decomposed the above signal by using the Gaussian, and damped sinusoidal dictionaries. After decomposing the signal and obtaining the best 16 atoms, we have then evaluated the first conditional spectral moment, and this is shown in Fig. 2. Because the mathematical expression for the signal in (22) is known, the true IF as well as the true WAIF (in general different from the IF for multicomponent signals) were both also derived and plotted in Fig. 2. Now although the first conditional spectral moment tends to be the IF for many bilinear time frequency distributions, if we compute this quantity based on MP, then it converges to the WAIF (as can be seen in Fig.2). This means that by using the MP distribution, we actually compute the WAIF for the decomposed signal. Finally, the amount of error between the true WAIF, and our estimate via the first conditional spectral moment, is dependent upon the kind of elementary function chosen and the number of algorithm iterations.



(b)

Fig. 2. The true WAIF (solid curve), the true IF (dashed curve) and the estimated first conditional spectral moment (dotted curve) based on MP signal decomposition for: (a) the Gaussian dictionary, and (b) the damped sinusoid dictionary. The number of iterations was 16.

4. DISCUSSION AND CONCLUSIONS

The interpretation of IF, defined as the derivative of the phase $\phi'(t)$ of a complex signal representation $a(t)e^{j\phi(t)}$, has been a subject of investigation and debate for years. One interpretation is that the IF is the average frequency at each time, because the first conditional spectral moment of many bilinear time frequency distributions equals the IF. The main paradox is that the instantaneous frequency often ranges beyond the spectral support of many signals. So the WAIF of the individual components may be a more useful quantity. That is fine as a definition, but it is generally difficult to use in practice because there is no systematic and general method for determining the components of a signal (i.e., the individual amplitudes and phases), which is itself a challenging problem. We have shown that the first conditional spectral moment, as computed from the MP distribution, is exactly the WAIF calculated from the decomposed signal. Two different known signals have been analysed and we have seen from simulations that what we compute as the first conditional spectral moment tends towards to the true WAIF, as the theory predicts. Finally, exactly what kind of elementary functions best extract the signal components is still an open question.

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