THE USE OF HILBERT TRANSFORM ON DSNR ANALYSIS OF STEP AND RIDGE EDGE DETECTION

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ABSTRACT

In this paper, the analytical expressions of Discriminative Signal to Noise Ratio (DSNR) of step-edge and ridge-edge via Hilbert Transform (HT) have been derived. Firstly we revisit those expressions evaluated for step-edge by using Canny's Edge Detector (CED) and Step Expansion Filter (SEF). Although SEF is optimal for step edges in terms of DSNR criterion, it is not optimal with respect to a variety of edge types. The matching filter has to be modified for different edge types. But a local energy measure obtained via HT can detect any kind of edges. The results obtained by HT are compared with the results of SEF and CED.

1. INTRODUCTION

It is well known that Hilbert transform is useful for generating the analytic signal, and it is used for single side modulation to save the bandwidth required in communication. However, it can be used for edge detection. In [1], the generalized radiant Hilbert transform is introduced and it is illustrated how to use it for edge detection.

Edge detection is the commonly used approach for detecting discontinuities in an image. Most of the edge detection methods are based on the derivative approach with linear filtering. Some of these filters are obtained from optimal criteria [2]. In [3], an optimal edge detection by expansion matching technique is presented. This matching technique optimizes a novel matching criterion called Discriminative Signal to Noise Ratio (DSNR) criterion. In [3], the authors claim that the DSNR criterion is better suited to evaluate matching in practical conditions than SNR since it considers as a noise, even the off-center response of the filter to the template itself. In this paper, optimal Step Expansion Filter (SEF) is compared with Canny Edge Detector (CED) for step edges.

In our study, we derive the analytical expressions of DSNR of step-edge and ridge-edge by the combination of CED and its Hilbert transform which we call Quadrature Pair Canny Edge Detector (QPCED) and compare with the results of (SEF) and (CED). The main role of using QPCED is that the energy measures are optimal with respect to a variety of edge types. The traditional filters used for edge detection have to be optimized for different kinds of edges. For instance, when the CED is applied to a ridge-edge rather than a step-edge, it produces two extrema in its output rather than one, each appearing at each side of the ridge-edge. If the filter is optimized for detecting ridge-edge, it will give spurious responses with edges. But a local energy measure obtained by using QPCED can detect any kind of edges. This energy measure is calculated from the square sum of QPCED outputs.

2. DISCRIMINATIVE SIGNAL-TO-NOISE RATIO

A novel matching criterion called DSNR is first presented in [3]. Since the off-center response to the template is considered as a noise, the optimal filter designed with respect to DSNR criterion generates highest response at the template's center. The DSNR is defined by the ratio of the power of the desired response to the total power of the undesired off-center response [3].

Consider the signal model:

$$s(x) = t(x) + \eta(x) \tag{1}$$

where $t(x)$ and $\eta(x)$ represent template and white noise with variance σ_η^2, respectively. Also we assume the noise $\eta(x)$ to be zero mean. The response of the filter with impulse response $h(x)$ to $s(x)$ is

$$g_s^h(x) = t(x) * h(x) + \eta(x) * h(x) = g_t^h(x) + g_\eta^h(x)$$

where $*$ represents convolution operator.

DSNR expression can be expressed as:

$$DSNR = 10\log \frac{[E\{g_s^h(0)\}]^2}{\int_{-\infty}^{\infty} [g_t^h(x)]^2 dx + E\{[g_\eta^h(x)]^2\}} = 10\log \frac{P_h}{N_h} \tag{2}$$

where P_h is the power of the desired response and N_h is the total power of undesired response.

3. DSNR ANALYSIS BY SEF AND CED

In this section we review the DSNR expressions evaluated by SEF and CED for step edge. The details of these expressions can be found in [3]. Also those expressions are derived for the ridge edge.

3.1 SEF

If the template, $t(x)$, in (1) is chosen as a step edge, $sgn(x)$, the optimal step edge detector, $\theta(x)$, is obtained from Wiener solution [3] such that

$$\theta(x) = \alpha e^{-|\frac{2x}{\sigma_\eta}|} sgn(x)$$

where α affects only the output magnitude and is redesigned to have unit magnitude response. The response of this filter to a step edge is

$$g_{sgn}^{\theta}(x) = sgn(x) * \theta(x) = \alpha \sigma_\eta e^{-|\frac{2x}{\sigma_\eta}|}$$

The response is constrained to be unit magnitude by $\alpha = 1/\sigma_\eta$.

The power of desired response of SEF in (2) becomes

$$P_\theta = [E\{g_s^\theta(0)\}]^2 = [g_{sgn}^\theta(0)]^2 = 1$$

and the total power of undesired response of SEF can be written as

$$\begin{aligned} N_\theta &= \int_{-\infty}^{\infty} [g_{sgn}^\theta(x)]^2 dx + E\{[g_\eta^\theta(x)]^2\} \\ &= \int_{-\infty}^{\infty} e^{-|\frac{4x}{\sigma_\eta}|} dx + E\{\int_{-\infty}^{\infty} \alpha\tau e^{-|\frac{2\tau}{\sigma_\eta}|} sgn(\tau)\eta(x-\tau)d\tau. \\ &\quad \int_{-\infty}^{\infty} \alpha\zeta e^{-|\frac{2\zeta}{\sigma_\eta}|} sgn(\zeta)\eta(x-\zeta)d\zeta\} \\ &= \frac{\sigma_\eta}{2} + \alpha^2 \int_0^\infty \int_0^\infty \tau\zeta e^{-\frac{2}{\sigma_\eta}(|\tau|+|\zeta|)} sgn(\tau)sgn(\zeta)R_{\eta\eta}(\tau-\zeta)d\tau d\zeta \end{aligned} \quad (3)$$

Since $R_{\eta\eta}(x) = \sigma_\eta^2\delta(x)$, (3) becomes

$$N_\theta = \frac{\sigma_\eta}{2} + \alpha^2\sigma_\eta^2 \int_{-\infty}^{\infty} \tau^2 e^{-|\frac{4\tau}{\sigma_\eta}|} d\tau = \frac{\sigma_\eta}{2} + \frac{\alpha^2\sigma_\eta^2}{16}$$

Consequently *DSNR* is obtained as [3]:

$$DSNR = 10\log(16/(\sigma_\eta^3 + 8\sigma_\eta)) \quad (4)$$

In the following, we consider the ridge edge as a template defined by

$$r(x) = u(x+X/2) - u(x-X/2)$$

where X is the width of a ridge edge. The Fourier transform (FT) of the ridge edge is obtained as

$$R(j\Omega) = 2\frac{\sin(\Omega X/2)}{\Omega}$$

The response of step expansion filter to a ridge edge is

$$g_r^\theta(x) = r(x) * \theta(x) = \alpha\frac{\sigma_\eta}{2} e^{\frac{-X}{\sigma_\eta}} [e^{\frac{2}{\sigma_\eta}x} - e^{\frac{-2}{\sigma_\eta}x}]$$

The power of desired response of SEF is obtained as

$$P_\theta = [E\{g_s^\theta(0)\}]^2 = [g_r^\theta(0)]^2 = 0 \quad (5)$$

3.2 CED

The Canny's edge detector is closely approximated by the 1^{st} derivative of Gaussian [4] given by

$$\phi(x) = \beta x e^{-x^2/2\sigma_c^2} \quad (6)$$

where β is an amplitude scale factor (approximately chosen to have unit magnitude response) and σ_c is the variance of the detector. The Fourier transform of $\phi(x)$ is obtained as

$$\Phi(j\Omega) = -j\Omega\sigma_c^3\beta\sqrt{2\pi}e^{-\Omega^2\sigma_c^2/2}$$

The response of this filter to a step edge and its Fourier transform are given by

$$\begin{aligned} g_{sgn}^\phi(x) &= sgn(x) * \phi(x) = -2\beta\sigma_c^2 e^{-x^2/2\sigma_c^2} \\ G_{sgn}^\phi(j\Omega) &= \sqrt{2\pi}\sigma_c e^{-\Omega^2\sigma_c^2/2} \end{aligned}$$

The response is constrained to be unit magnitude by $\beta = -1/2\sigma_c^2$. σ_c is determined such that the total power of the undesired response is minimized. Therefore the maximum possible *DSNR* is obtained as

$$DSNR = 10\log(\sqrt{\frac{2}{\pi\sigma_\eta^2}}) \quad (7)$$

The response of CED to a ridge and its FT are

$$\begin{aligned} g_r^\phi(x) &= r(x) * \phi(x) \\ &= -\beta\sigma_c^2[e^{-(X/2+x)^2/2\sigma_c^2} - e^{-(-X/2+x)^2/2\sigma_c^2}] \\ G_r^\phi(j\Omega) &= R(j\Omega)\Phi(j\Omega) \\ &= -j2\sqrt{2\pi}\sigma_c^3\beta\sin(\frac{\Omega X}{2})e^{-\Omega^2\sigma_c^2/2} \end{aligned}$$

The power of desired response of CED becomes

$$P_\phi = [E\{g_s^\phi(0)\}]^2 = [g_r^\phi(0)]^2 = 0 \quad (8)$$

and the total power of the undesired response of the CED is obtained as

$$\begin{aligned} N_\phi &= \int_{-\infty}^{\infty} [g_r^\phi(x)]^2 dx + E\{[g_\eta^\phi(x)]^2\} \\ &= \int_{-\infty}^{\infty} [g_r^\phi(x)]^2 dx + \frac{\sigma_\eta^2}{8\sigma_c}\sqrt{\pi} \end{aligned} \quad (9)$$

By using Parsevals relation, (9) can be written as

$$\begin{aligned} N_\phi &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_r^\phi(j\Omega)|^2 d\Omega + \frac{\sigma_\eta^2}{8\sigma_c}\sqrt{\pi} \\ &= 4\sigma_c^5\beta^2\sqrt{\pi} + \frac{\sigma_\eta^2}{8\sigma_c}\sqrt{\pi} \end{aligned} \quad (10)$$

4. DSNR ANALYSIS FOR QPCED

In this section, we derive the DSNR expressions of step and ridge edges by the combination of CED and its Hilbert transform which we call Quadrature Pair CED (QPCED) since their FTs have same magnitude response but differ by $\pi/2$ in phase. The expression of Canny's edge detector is given in (6). The Hilbert transform of CED,$\gamma(x)$, and its FT,$\Gamma(j\Omega)$, are obtained as

$$\gamma(x) = \frac{1}{\pi x} * \phi(x) = \int_{-\infty}^{\infty} \frac{1}{\pi\tau}.\beta(x-\tau)e^{-(x-\tau)^2/2\sigma_c^2} d\tau$$

$$\Gamma(j\Omega) = -jsgn(j\Omega)\Phi(j\Omega) = -sgn(j\Omega)\Omega\sigma_c^3\beta\sqrt{2\pi}e^{-\Omega^2\sigma_c^2/2} \quad (11)$$

4.1 Step Edge

The response of the filter whose the impulse response is $\gamma(x)$ to a step edge and its FT are obtained as

$$\begin{aligned} g_{sgn}^\gamma(x) &= sgn(x) * \gamma(x) = g_{sgn}^\phi(x) * \frac{1}{\pi x} \\ G_{sgn}^\gamma(j\Omega) &= -jsgn(j\Omega)G_{sgn}^\phi(j\Omega) \\ G_{sgn}^\gamma(j\Omega) &= -j\sqrt{2\pi}\sigma_c sgn(j\Omega)e^{-\Omega^2\sigma_c^2/2} \end{aligned} \quad (12)$$

The power of desired response of QPCED is

$$\begin{aligned} P_{\phi+\gamma} &= [\{g_s^\phi(0)\}]^2 + [\{g_s^\gamma(0)\}]^2 = [g_{sgn}^\phi(0)]^2 + [g_{sgn}^\gamma(0)]^2 \\ &= P_\phi + P_\gamma \end{aligned}$$

Since $g_{sgn}^\gamma(0) = (1/2\pi)\int_{-\infty}^{\infty} G_{sgn}^\gamma(j\Omega)d\Omega$, P_γ becomes $P_\gamma = [(1/2\pi)\int_{-\infty}^{\infty} G_{sgn}^\gamma(j\Omega)d\Omega]^2 = 0$. Therefore $P_{\phi+\gamma}$ is equal to P_ϕ which is 1 in the case of choosing β as $-1/2\sigma_c^2$.

The total power of the undesired response of the QPCED is written as

$$\begin{aligned} N_{\phi+\gamma} &= N_\phi + \int_{-\infty}^{\infty} [g_{sgn}^\gamma(x)]^2 dx + E\{[g_\eta^\gamma(x)]^2\} \\ &= N_\phi + N_\gamma \end{aligned} \quad (13)$$

By using Parseval's relation, (13) can be written as

$$N_{\phi+\gamma} = N_\phi + \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_{sgn}^\gamma(j\Omega)|^2 d\Omega + \frac{\sigma_\eta^2}{2\pi}\int_{-\infty}^{\infty} |\Gamma(j\Omega)|^2 d\Omega \quad (14)$$

Substituting (10), (11) and (12) into (14), we obtain

$$N_{\phi+\gamma} = 2\sqrt{\frac{\pi}{2}}\sigma_\eta \quad (15)$$

and consequently $DSNR$ is obtained as

$$DSNR = 10\log(\sqrt{\frac{1}{2\pi\sigma_\eta^2}}) \quad (16)$$

4.2 Ridge Edge

The response of the $\gamma(x)$ to a ridge edge and its FT are

$$\begin{aligned} g_r^\gamma(x) &= r(x) * \gamma(x) = g_r^\phi(x) * \frac{1}{\pi x} \\ G_r^\gamma(j\Omega) &= -jsgn(j\Omega)G_r^\phi(j\Omega) \\ &= -2\sqrt{2\pi}\sigma_c^3\beta sgn(j\Omega)\sin(\frac{\Omega X}{2})e^{-\Omega^2\sigma_c^2/2} \end{aligned}$$

The power of desired response of QPCED is

$$\begin{aligned} P_{\phi+\gamma} &= [\{g_s^\phi(0)\}]^2 + [\{g_s^\gamma(0)\}]^2 = [g_r^\phi(0)]^2 + [g_r^\gamma(0)]^2 \\ &= P_\phi + P_\gamma = P_\gamma \end{aligned}$$

Since $g_r^\gamma(0) = (1/2\pi)\int_{-\infty}^{\infty} G_r^\gamma(j\Omega)d\Omega$, P_γ can be written as

$$\begin{aligned} P_\gamma &= [\frac{1}{2\pi}\int_{-\infty}^{\infty} G_r^\gamma(j\Omega)d\Omega]^2 \\ &= (\frac{\sigma_c^3\beta c}{j\sqrt{\pi}})^2[\int_0^\infty e^{-(\Omega a-b)^2}d\Omega - \int_0^\infty e^{-(\Omega a+b)^2}d\Omega]^2 \end{aligned} \quad (17)$$

where $a = \frac{\sigma_c}{\sqrt{2}}$, $b = j\frac{X}{4}\frac{\sqrt{2}}{\sigma_c}$, $c = e^{-\frac{X^2}{8}\frac{1}{\sigma_c^2}}$. Let $(\Omega a - b) = u$ and $(\Omega a + b) = v$. Therefore (17) becomes

$$\begin{aligned} P_\gamma &= (\frac{\sigma_c^3\beta c}{ja\sqrt{\pi}})^2[(\int_{-b}^{\infty} e^{-u^2}du - \int_b^\infty e^{-v^2}dv)]^2 \\ &= (\frac{\sigma_c^3\beta c}{ja\sqrt{\pi}})^2[(1 - 2\int_b^\infty e^{-v^2}dv)]^2 \\ &= (\frac{\sigma_c^3\beta c}{ja\sqrt{\pi}})^2[1 - 4\int_b^\infty e^{-v^2}dv + 16\pi e^{-2b^2}] \end{aligned} \quad (18)$$

In (18), β has to be determined in terms of σ_c such that unit magnitude power is obtained. We use the approximation of $\int_b^\infty e^{-v^2}dv$ in (18) given by

$$\int_b^\infty e^{-v^2}dv \cong \frac{1}{(1-\frac{1}{\pi})b + \frac{1}{\pi}\sqrt{b^2 - 2\pi}}e^{-b^2/2}$$

Therefore β is chosen approximately as

$$\beta^2 \cong \frac{1}{(\frac{\sigma_c^3 c}{ja\sqrt{\pi}})^2[1 - 4\frac{1}{(1-\frac{1}{\pi})b+\frac{1}{\pi}\sqrt{b^2-2\pi}}e^{-b^2/2} + 16\pi e^{-2b^2}]} \quad (19)$$

The total power of the undesired response of QPCED is

$$N_{\phi+\gamma} = N_\phi + \int_{-\infty}^{\infty} [g_r^\gamma(x)]^2 dx + E\{[g_\eta^\gamma(x)]^2\} \quad (20)$$

Substituting (10) and the result of $E\{[g_\eta^\gamma(x)]^2\}$ obtained in (14) into (20) and by using Parseval's relation, we obtain

$$N_{\phi+\gamma} = 4\sigma_c^5\beta^2\sqrt{\pi} + 2\sqrt{2\pi}\sigma_c^5\beta^2[1 - e^{\frac{-X^2}{4\sigma_c^2}}] + \frac{\sigma_\eta^2}{4\sigma_c}\sqrt{\pi} \quad (21)$$

After substituting (19) into (21), σ_c has to be determined such that the total power of undesired response is minimized by $\partial N_{\phi+\gamma}/\partial\sigma_c = 0$. We obtain the results numerically by using the steepest descent method presented in [5]. σ_c which minimizes $N_{\phi+\gamma}$ is evaluated for different σ_η^2 and consequently $DSNR$ is obtained for these values as

$$DSNR = 10\log\frac{1}{N_{\phi+\gamma}} \quad (22)$$

5. SIMULATION RESULTS

In this section, the simulation results for DSNR performance versus SNR obtained by SEF, CED and QPCED are presented.

In Fig. 1, the amount of additive white noise to the step edge is varied from 27dB to -5dB. These plots are based on the DSNR expressions given in (4), (7) and (16). SEF, CED and QPCED are redesigned for each noise level according to these expressions. It is seen that the SEF outperforms the CED and QPCED for the step edge. Although the SEF is optimal for the step edge in terms of DSNR criteria, it is not optimal with respect to a variety of edge types. The matching filter has to be modified for different edge types. As obtained in (5) and (8), the powers of desired responses of SEF and CED to a ridge are zero. The DSNR performance of QPCED for the ridge is illustrated in Fig. 2. In this figure the DSNR which is numerically obtained by using steepest descent method versus ridge SNR for different widths is shown. As can be seen in this figure, QPCED outperforms for the width $X = 1$. The power of white noise is varied between 34dB and -5dB. σ_c is determined such that $N_{\phi+\gamma}$ given in (21) is minimized for these values of noise power and then DSNR shown in Fig. 2 is obtained from these values of σ_c.

Fig. 3 and 4 depict the SEF, CED and QPCED responses to the noisy step edge with an SNR of 10dB, respectively. It can be seen that a peak is obtained as a response of SEF, CED and QPCED at the edge location. Also Fig. 4 illustrate that QPCED perform better than the SEF and CED for the noisy ridge edge with an SNR of 10dB.

6. CONCLUSION

We have presented the DSNR analysis of step and ridge edge detection using QPCED and compare with the results of SEF and CED. Although SEF is optimal for the step-edge in terms of DSNR criteria, it is possible to detect the step edge by CED and QPCED .

As a part of future work, DSNR analysis could be extended into 2D case. Further analysis could also include Monte Carlo simulations to characterize the performance of our proposed approach.

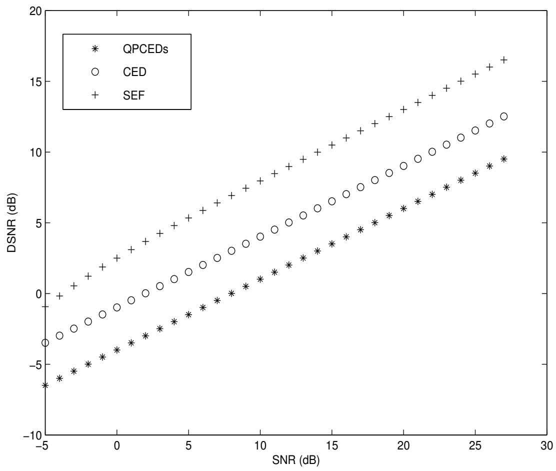

Figure 1: Plot of DSNR versus step SNR.

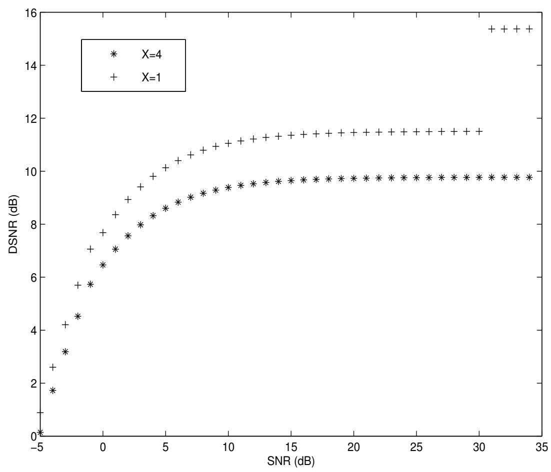

Figure 2: Plot of DSNR versus ridge SNR for different widths.

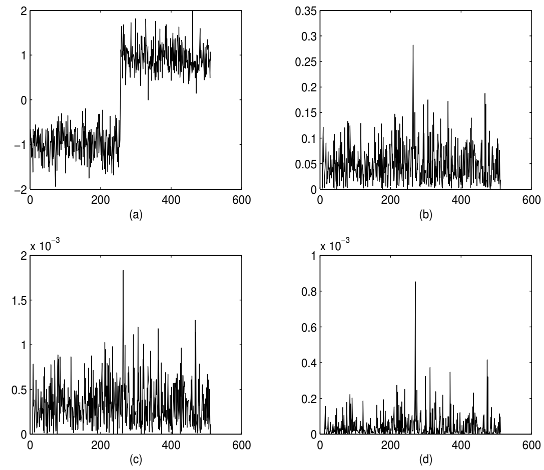

Figure 3: (a) Noisy step input. SNR=10dB. (b) SEF response. (c) CED response. (d) QPCED response.

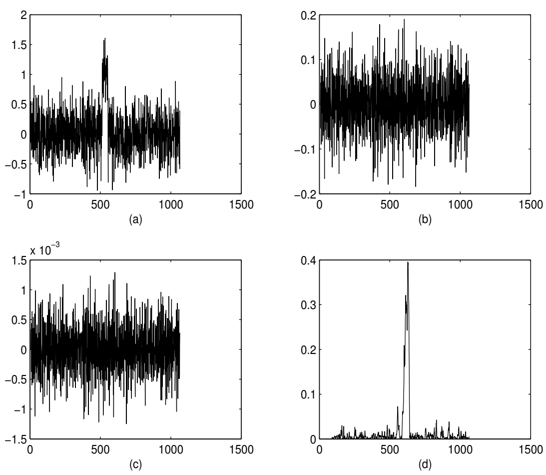

Figure 4: (a) Noisy ridge input. SNR=10dB. (b) SEF response. (c) CED response. (d) QPCED response.

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