ON THE REPRESENTATION ERROR OF DIGITIZED SIGNALS

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ABSTRACT

Although the origin of many sources of information is analogue or continuous-time, due to practical considerations signal representation is usually based on discrete-time basis functions. These basis functions are in many cases sampled versions of harmonic signals (Fourier), Gabor, wavelets and the like. In this paper, we examine the accuracy of this approach, where the calculation of the representation coefficients is performed using digital inner product rather than analogue. By interpreting the sampling process as a bounded linear operator, we analyze the difference between the analogue domain inner product and the result one may get in the digital domain. We consider both the one-dimensional and two-dimensional cases, and several applicable examples are given.

1. INTRODUCTION

Signal processing applications are concerned mainly with digital data, although the origin of many sources of information is analogue, such as speech and audio, optics, radar, sonar, velocities, forces, biomedical signals and many more. For many situations it is well known that the set of functions $\{\mathrm{sinc}\,(t/T-n)\}_n$ constitutes an orthogonal basis for the space of π/T - band limited functions [1], and that the corresponding representation coefficients are then simply obtained as the uniformly spaced samples of the original signal. Most signals, however, are not band limited. Thus, considering their sampled version as a representation scheme introduces errors. For this reason mainly, alternative basis functions such as Gabor functions, wavelets, and other orthogonal function-sets are often used instead [2,3,4].

Finding representation coefficients for these alternative basis functions involves L_2 inner-product calculations within the analogue domain, rather than simply consider the sampled version of the signal itself as in the case of band limited functions. This in turn is in most cases difficult to implement, and even impossible in applications where the only data available are the sampled version of the signal itself.

To overcome this difficulty, it may be naively assumed that this L_2 inner product could be reasonably well approximated by an l_2 one, i.e.,:

(1) $$\langle f,\varphi\rangle \cong T\cdot\sum_n f(nT)\cdot\varphi(nT),$$

where $f(t)$ is the original signal and $\varphi(t)$ is a known (basis) function. However, with no prior knowledge of the original signal $f(t)$ beyond its samples, no bound on this approximation error is presently available.

The question raised in this work is whether the sampling process keeps algebraic relations, shared within the analogue domain, intact. This question is of interest in signal processing applications. To investigate this task of signal representation, we consider the operation widely used in vector representation, the inner product. More specifically, we search for alternative approximation schemes for this inner product within the digital domain.

2. THE PROBLEM

We address the following problem (Figure 1): Given a function $\varphi(t)\in L_2$, what is the approximation error of the inner product of $\langle f,\varphi\rangle$, if only the samples of $f(t)$ are available?

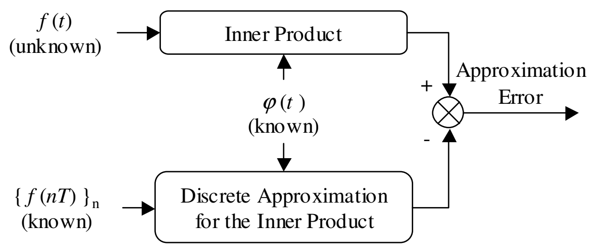


Figure 1: ***Statement of the problem - given a function $\varphi(t)$, what is the approximation error of the inner product of $\langle f,\varphi\rangle$, if only the samples of $f(t)$ are avaiable?***

3. SAMPLING AS A LINEAR OPERATOR

The analogue sources that interest signal processing practitioners have finite energy. Thus, representation coefficients are extracted by applying an L_2 inner product:

(2) $$\langle f,\varphi\rangle_{L_2} = \int f(t)\cdot\varphi(t)\,dt\,,$$

where for simplicity, only real functions are considered. Approximating both f and φ by piecewise constant functions can be interpreted as approximating this L_2 inner product by an l_2 inner product of two finite-energy sequences:

(3) $$\langle f,\varphi\rangle_{L_2} \cong \langle f(nT), T\cdot\varphi(nT)\rangle_{l_2}\,.$$

This interpretation, however, should be done with much prudence; sampling a function of L_2 does not necessarily yield a sequence of l_2. Blu and Unser [5] have shown, nevertheless, that sampling a Sobolev function [6] of order one, i.e. $f(t)$, $f'(t) \in L_2$, would always yield a sequence having finite energy. The importance of this result resides in the fact that the sampling process can now be considered as a linear bounded operator [7] acting on a Sobolev space of an arbitrary order to yield an l_2 sequence. A Sobolev space of order n (denoted here W_2^n) consists of all finite energy functions having at least n finite energy derivatives. It is a Hilbert space considering the following inner product:

(4)
$$\langle f,\varphi\rangle_{W_2^n} = \sum_{i=0}^{n}\left\langle f^{(i)},\varphi^{(i)}\right\rangle_{L_2} = \frac{1}{2\pi}\int F(\omega)\overline{\Phi(\omega)}\left(1+\cdots+\omega^{2n}\right)d\omega ,$$

where F and Φ denote for the Fourier transform of f and φ respectively. Sobolev functions of an arbitrary order are dense in L_2, therefore, restricting our analysis to such functions still maintains generalization of the results.

We start our analysis with the introduction of two lemmas:

Lemma 1: The sampling operator S_T is given by,

(5)
$$S_T : W_2^n \to l_2$$
$$S_T f = \sum_n \langle f(t), u(t-nT)\rangle_{W_2} \cdot e_n \quad ,$$

where $\{e_n\}$ is the standard basis of l_2, and $u(t)$ is the inverse Fourier transform of ,

(6)
$$U(\omega) = \frac{1}{1+\omega^2+\omega^4+\cdots+\omega^{2n}} .$$

Lemma 2: The adjoint operator of S_T, namely S_T*, is given by,

(7)
$$S_T^* : l_2 \to W_2^n$$
$$\left(S_T^* b\right)(t) = \sum_n b[n]\cdot u(t-nT)$$

where $u(t)$ is given in Lemma 1.
The proof of the above 2 lemmas is given in [8].

4. INNER PRODUCT: INTERTWINING RELATIONS OF L_2, l_2 & W_2

It has been shown [8] that both L_2 and l_2 inner products can be expressed as inner products of a Sobolev space of order n, which leads to the following result (Figure 2).

Lemma 3: Let $\varphi(t) \in L_2$ be a known function, and let $b[n] \in l_2$ be a known sequence. Then, $\forall f \in W_2^n$:

(8)
$$\langle f,\varphi\rangle_{L_2} - \langle S_T f, b\rangle_{l_2} = \left\langle f, \varphi^* - S_T^* b\right\rangle_{W_2^n} .$$

S_T is the uniform sampling operator given in Lemma 1, S_T^* - its adjoint given in Lemma 2 and φ* is given by,

(9)
$$\varphi^*(t) = \varphi(t) * u(t),$$

where $u(t)$ is given in Lemma 1, as well.
The proof is given in [8].

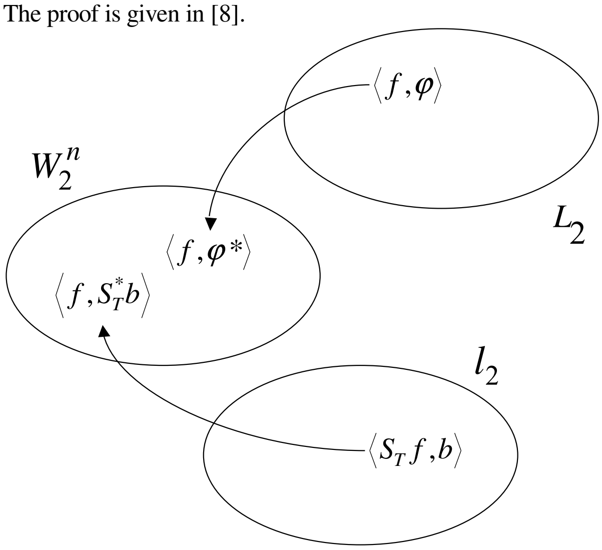


Figure 2: ***Intertwining relations of L_2-, l_2- & W_2^n inner products. φ is a known function of L_2, b is a known sequence of l_2. f is an arbitrary Sobolev function of an arbitrary order to be uniformly sampled. The inner product of L_2 has a corresponding representation in W_2^n, and the same holds for the l_2 inner product.***

5. SAMPLING EFFECTS UPON L_2 INNER PRODUCT

The following theorem is the basis to our analysis of the approximation error.

Theorem 1:
Let $\varphi(t) \in W_2^1$ be a known function. Given a sampling interval T, the following relation holds for any Sobolev function $f(t) \in W_2^n$:

(10)
$$\left|\langle f,\varphi\rangle_{L_2} - T\langle S_T f, S_T\varphi\rangle_{l_2}\right| \le B\cdot\|f\|_{L_2} ,$$

where S_T is the uniform sampling operator with interval T, B is given by,

(11)
$$B = \frac{\left\|\varphi(t)*u(t) - T\sum_n \varphi(nT)\cdot u(t-nT)\right\|_{W_2^n}^2}{\left\|\varphi(t)*u(t) - T\sum_n \varphi(nT)\cdot u(t-nT)\right\|_{L_2}} ,$$

and $u(t)$ is the inverse Fourier transform of ,

(12)
$$U(\omega) = \frac{1}{1+\omega^2+\omega^4+\cdots+\omega^{2n}} .$$

The proof is given in [8].

Example 1: Sampling Hermite functions.

The Hermite functions constitute an orthonormal basis in L_2. Thus, the representation coefficients are to be found by calculating $a_k = \langle f, \varphi_k \rangle_{L2}$, where φ_k is the Hermite function of order k. Having both the sampled versions of f and φ_k, a_k would be approximated by,

$$\hat{a}_k = T \cdot \langle S_T f, S_T \varphi_k \rangle_{l_2} = T \cdot \sum_n f(nT) \cdot \varphi_k(nT) \tag{13}$$

This in turn would yield an approximation error, upper bounded by $B \cdot \|f\|_{L2}$. Some bounds are shown for example in Figure 3 for the first Hermite function (a Gaussian).

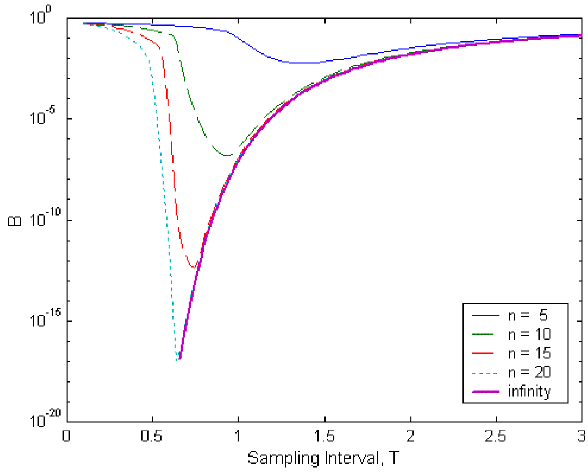


Figure 3: *Upper bounds for the approximation of $\langle f, \psi_k \rangle$ by their corresponding sampled versions. Here ψ_k is a Hermite function of order k = 0 (a Gaussian). The upper bound for this error is given by $B \cdot \|f\|$. Shown are upper bounds where the admissible functions, f, are Sobolev functions of several orders* (**5,10,15,20** *and infinity).*

6. SAMPLING EFFECTS UPON L_2 INNER PRODUCT – THE TWO DIMENSIONAL CASE (IMAGES)

Introducing the two-dimensinal Sobolev space, it consistes of the following inner product [6]:

$$\langle f, g \rangle_{W_2^n} = \sum_{0 \le |\alpha| \le n} \langle D^\alpha f, D^\alpha g \rangle_{L_2}, \tag{14}$$

where $\alpha = (\alpha_1, \alpha_2)$ is a 2-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha = \partial^{\alpha 1}/\partial x \cdot \partial^{\alpha 2}/\partial y$. The function $u(t)$ then, defined in Lemma 1, corresponds now to the inverse Fourier transform of $U(u,v)$, given by the following equation (few examples are given in Table 1):

$$U(u,v) = \frac{1}{\sum_{0 \le |\alpha| \le n} \left(u^{\alpha_1} \cdot v^{|\alpha| - \alpha_1} \right)^2}. \tag{15}$$

Sobolev order, *n*	*U(u,v)*
1	$\left(1+u^2+v^2\right)^{-1}$
2	$\left(1+v^2+v^4+u^2+(uv)^2+u^4\right)^{-1}$
3	$\left(1+v^2+v^4+v^6+u^2+(uv)^2+ + \left(uv^2\right)^2+u^4+\left(u^2v\right)^2+u^6\right)^{-1}$

Table 1: ***U(u,v)**, shown here for several Sobolev orders* (**n=1, 2, 3**)*, describes the sampling functional in terms of a Sobolev inner product for two dimensional signals, i.e.* $f(x_0,y_0) = \langle f, u(x-x_0, y-y_0) \rangle_{W2^n}$.

This derivation of $u(x,y)$, in turn, enables one to analyze the sampling effects on the inner product with regard to images;

Theorem 2:

Let $\varphi(x,y) \in W_2^1$ be a known function. Given a sampling interval T, the following relation holds for any Sobolev function $f(t) \in W_2^n$:

$$\left| \langle f, \varphi \rangle_{L_2} - T^2 \cdot \langle S_T f, S_T \varphi \rangle_{l_2} \right| \le B \cdot \|f\|_{L_2}, \tag{16}$$

where S_T is the uniform sampling operator with an interval T. B is then given by,

$$\frac{\left\| \varphi(x,y) * u(x,y) - T^2 \cdot \sum_{n,m} \varphi[nT, mT] \cdot u(x-nT, y-mT) \right\|^2_{W_2^n}}{\left\| \varphi(x,y) * u(x,y) - T^2 \cdot \sum_{n,m} \varphi[nT, mT] \cdot u(x-nT, y-mT) \right\|_{L_2}} \tag{17}$$

The proof is given in [8]

Example 2: Suppose one wishes to determine whether two images are different or not. i.e., it is required to approximate the representation coefficients according to a set of basis images. Assuming that one of these basis images is the Gaussian, its representation coefficient can be calculated for various sampling intervals, as shown in Figure 4. The key point, however, is that utilizng the abovementioned results; one can also extract the maximum potential approximation error induced by the sampling process in advance. Based on that information, a proper decision can then be made. It is evident from Figure 4, that a sampling interval of $T = 1$ is insufficient for approximating the repersentation coefficient of the original images with regard to the Gaussian image. $T = 0.5$ is however sufficient, and there is no need to consider smaller sampling intervals such as $T = 0.1$.

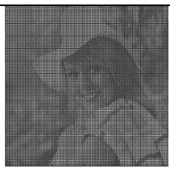
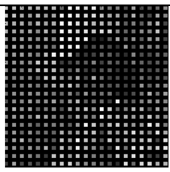
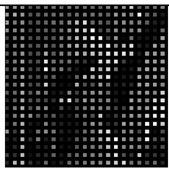
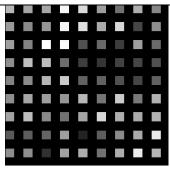
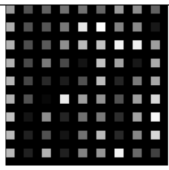
Sampling Interval	Images		Inner Product with a Gaussian image	Maximum Potential Error
Original			-	-
***T* = 0.1**			15.18 vs. 16.18	± 0.268
***T* = 0.5**			14.64 vs. 15.73	± 0.61
***T* = 1**			14.89 vs. 15.68	± 0.857

Figure 4: ***An example, utilizing Theorem 2. It is evident, that a sampling rate of T = 1 is insufficient for approximating the representation coefficient of the original images with regard to the Gaussian image. T = 0.5 is however sufficient, and there is no need to consider smaller sampling intervals such as T = 0.1.***

7. CONCLUSIONS

The results presented in this paper are applicable to digital signal and image processing systems, in which proper representation of time-continuous signals is required, while having only the sampled version of originally analogue signals. It also enables one to determine the representation error induced by the sampling process of non band-limited signals in advance. Our analysis provides also a means for determing the sufficient sampling rate when a certain level of an approximation error is imposed.

Furthermore, our vector-like interpretation suggests an alternative discrete approximation scheme for the inner product, which utilizes rather a different sequence than the sampled version of the basis function $\varphi(t)$. This idea is under further investigation.

8. ACKNOWLEDGMENTS

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