MINIMUM-PHASE FIR FILTER DESIGN USING REAL CEPSTRUM

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ABSTRACT

The real cepstrum is used to design an arbitrary length minimum-phase FIR filter from a mixed-phase sequence. There is no need to start with the odd-length equiripple linear-phase sequence first. Neither phase-unwrapping nor root-finding is needed. Only two FFTs and an iterative procedure are required to compute the filter impulse response from real cepstrum; the resulting magnitude response is exactly the same with the original sequence.

1. INTRODUCTION

In many low delay applications of FIR filters design such as data communication system, linear phase characteristic is not necessary and, minimum phase design can preserve desired magnitude response and has the advantage of minimum delay over other counterparts with the same magnitude response.

There has many methods been developed to design minimum-phase FIR filters, especially the one proposed by Herrmann and Schuessler [1]. It starts with an odd-length linear-phase equiripple FIR filter and shifts it up by one-half the stop band's peak-to-peak ripple, resulting in second-order zeros on the unit circle. The zeros inside the unit circle and a simple zero out of each pair of double zeros on the unit circle are then retained to obtain the minimum-phase filter with half the degree. However, the difficulty of root-finding procedure limits this method and the magnitude response becomes approximately square root of the original one. Therefore, later researches resorted to other methods to avoid root-finding procedure. Mian and Nainer [2] utilized the complex cepstrum to extract the minimum-phase component. In this method, only two FFTs are required, but cumbersome process of phase-unwrapping is required. To avoid phase-unwrapping, Pei and Lu [3] applied differential cepstrum to design the equiripple minimum-phase FIR filter, but 3 FFTs are required. Rather than using cepstrum, an approach based on Newton-Raphson iterative algorithm [4] was recently recommended to find the minimum-phase spectral factor of polynomials. The work emphasizes on its better accuracy than that could be obtained by root-finding when there's no zeros on the unit circle. If there are indeed double zeros on the unit circle, however, precision will be lost.

Among the several methods mentioned above, all are to design a linear-phase equiripple FIR filter first. If this linear phase filter is not equiripple in the stop band, we cannot merely shift up its transfer function to get the sequence with double zeros for its z-transform. Moreover, the minimum phase filter magnitude response becomes square root of the original one by keeping half the zeros on and inside the unit circle. Recently, a different approach based on root moments was proposed to design minimum-phase FIR filters [6], which preserves the same magnitude response. But it needs to start from a linear phase FIR filter, due to the complex conjugate relation between its zeros. Moreover, we need to select a proper radius of integration contour in advance to calculate moments accurately. From the previous works of Mian and Nainer [2], we can extend it and avoid phase-unwrapping by using real cepstrum. This benefits from the problem itself, that is, constructing the minimum-phase component from its magnitude. It is known that a minimum-phase sequence's magnitude determines its phase. Through several deductions, we will find that real cepstrum determines a sequence's minimum-phase component. Moreover, in our works, the minimum-phase filter will retain the original magnitude response exactly.

In the following sections, we first discuss several basic related concepts. Next, the formal steps for minimum-phase sequence construction using real cepstrum will be summarized. Furthermore, we refer to this method in an alternative viewpoint by treating it as passing the original sequence through an allpass filter. Finally, several design examples are shown and illustrated the effectiveness of this approach.

2. BASIC CONCEPTS ON CEPSTRUM

A. Complex Cepstrum and Real Cepstrum

Let $h(n)$ be a real sequence with $H(e^{j\omega})$ as its Fourier transform. Its complex cepstrum $\hat{h}(n)$ and real cepstrum $\hat{c}(n)$ are defined as

$$\hat{H}(e^{j\omega}) = \log\left[H(e^{j\omega})\right] = \log\left|H(e^{j\omega})\right| + j\arg[H(e^{j\omega})] \tag{1}$$

$$\hat{C}(e^{j\omega}) = \Re e\left\{\hat{H}(e^{j\omega})\right\} = \log\left|H(e^{j\omega})\right| \tag{2}$$

$$\hat{h}(n) = F^{-1}\left\{\hat{H}(e^{j\omega})\right\} \tag{3}$$

$$\hat{c}(n) = F^{-1}\left\{\hat{C}(e^{j\omega})\right\} \tag{4}$$

where F^{-1} denotes the inverse Fourier transformation.

Note that in (1) and (3), to compute complex cepstrum, we need to perform logarithm on a complex number. The imaginary part of the complex logarithm must be continuous and without its linear-phase term to avoid ambiguity. To achieve this, we can first compute the principle value of the phase (between -π and π), then unwrap the phase to a continuous one, and remove the linear-phase term.

B. Properties of Minimum/Maximum-Phase Sequence and Its Complex Cepstrum

From [5], there are two useful properties.

a. If $h(n)$ is a minimum-phase sequence, $\hat{h}(n)$ will be a causal sequence. That is, $\hat{h}(n)=0$ for $n<0$.
b. If $h(n)$ is a maximum-phase sequence, $\hat{h}(n)$ will be an anti-causal sequence. That is, $\hat{h}(n)=0$ for $n>0$

C. Explicit Formula between Minimum-Phase Sequence and Its Complex Cepstrum

An arbitrary sequence $h(n)$ and its complex cepstrum $\hat{h}(n)$ has an implicit recursive relation [5] as

$$h(n)=\begin{cases} e^{\hat{h}(0)} & n=0 \\ \sum_{k=-\infty}^{\infty}\left(\frac{k}{n}\right)\hat{h}(k)h(n-k) & n\neq 0 \end{cases} \tag{5}$$

If $h(n)$ is a finite minimum-phase sequence, the above summation can be reduced to finite terms as

$$h(n)=\begin{cases} e^{\hat{h}(0)} & n=0 \\ \sum_{k=0}^{n}\left(\frac{k}{n}\right)\hat{h}(k)h(n-k) & n>0 \end{cases} \tag{6}$$

D. Reconstruction of a Causal Sequence from its Even Part

If $h(n)$ is a causal sequence, it can be recovered simply by its even part sequence $h_e(n)$

$$h(n)=h_e(n)\,u_+(n), \tag{7}$$

where $u_+(n)=\begin{cases} 0 & n<0 \\ 1 & n=0 \\ 2 & n>0 \end{cases}$

E. Fourier Transform Pair between Time- and Frequency-Domain

Let $h_e(n)$ and $h_0(n)$ are the even and odd parts of sequence $h(n)$. $H_R(e^{j\omega})$ and $H_R(e^{j\omega})$ are the real and imaginary parts of its Fourier transform, respectively. Now, if $h(n)$ is a real sequence, we have the following Fourier transform relations :

$$F\{h_e(n)\}=H_R(e^{j\omega}) \tag{8}$$

$$F\{h_o(n)\}=jH_I(e^{j\omega}) \tag{9}$$

3. CONSTRUCTION OF MINIMUM-PHASE SEQUENCE

Let $h_{\min}(n)$ denote the minimum-phase counterpart of $h(n)$, $\hat{h}_{\min}(n)$ and $\hat{c}_{\min}(n)$ be the complex cepstrum and the real cepstrum of $h_{\min}(n)$, respectively.

From the above concept, if we want to extract the minimum-phase part from a mixed-phase sequence $h(n)$, we can simply drop the non-causal part of $\hat{h}(n)$ to acquire $\hat{h}_{\min}(n)$, and calculate $h_{\min}(n)$ directly using (6). Calculating complex cepstum $\hat{h}(n)$ from $h(n)$, however, is concerned with taking logarithm on complex number, where phase unwrapping is needed. We can skip the complicated procedure of phase unwrapping by adopting its real cepstrum, which does not involve phase unwrapping. The reason we can directly use real cepstrum comes from (7) and (8) as shown in Fig. 1(a) and (b). Note that $\hat{C}(e^{j\omega})$ is the real part of $\hat{H}(e^{j\omega})$ from (2). Because the real cepstrum $\hat{c}(n)$ is the inverse Fourier transform of $\hat{C}(e^{j\omega})$, we find that real cepstrum is exactly the even part sequence of complex cepstrum using (8). If we further combine this corollary with (7), now we know we can reconstruct $\hat{h}_{\min}(n)$ from $\hat{c}(n)$ using (7), rather than from $\hat{h}(n)$.

If $H(z)$ has its zeros on the unit circle, the region of convergence of $\log[H(z)]$ cannot include the unit circle. From this computational point of view, we should avoid the zeros existing on the unit circle. But in practice, it's often to design digital filters with some zeros on the unit circle in z-domain. We can overcome this problem by selecting a different contour C slightly inside unit circle while computing $\hat{c}(n)$ from $\hat{C}(z)$ [5]. This can be achieved equivalently by first multiplying the input $h(n)$ with an exponential sequence as

$$w(n)=\begin{cases} \alpha^n h(n) & \text{n = 0,...., N -1} \\ 0 & \text{n} = N,.....L-1 \end{cases} \quad with\ L=1024>>8N \tag{10}$$

where $\alpha<1$ and $\alpha\cong 1$. This step will cause the radius of its zeros scaled down by the factor α , i.e. moving its zeros slightly inside the unit circle.

Besides, even though the sequence is finite, its cepstrum sequence is still infinite [5]. Computationally, aliasing effect will occur. To reduce the aliasing effect, we must append $w(n)$ with several trailing zeros as in (10).

Now we summarize the overall steps for constructing the minimum-phase sequence $h_{\min}(n)$ from any mixed-phase sequence $h(n)$ as follows :

1. Choose $\alpha<1$ *and* $\alpha\cong 1$ to move the zeros slightly inside on the unit circle.
2. Perform L-point $(\text{FFT})_\text{L}$ on $\alpha^n h(n)$, $n=0,1,\ldots,(N-1)$, to get $H_\alpha(k)$, $k=0,1,\ldots,(L-1)$, $L=1024>>8N$.
3. Perform $(\text{IFFT})_\text{L}$ on $\log|H_\alpha(k)|$ to get $\hat{c}_\alpha(n)$.
4. Construct $\hat{h}_{\min,\alpha}(n)$ from $\hat{c}_\alpha(n)$ using (7).
5. Compute $h_{\min,\alpha}(n)$ from $\hat{h}_{\min,\alpha}(n)$ using (6).
6. Rescale $h_{\min,\alpha}(n)$ to get $h_{\min}(n)=h_{\min,\alpha}(n)\alpha^{-n}$.

4. ALLPASS FILTERING VIEWPOINT

In this section, we provide another viewpoint on the above works. In fact, the proposed process in the previous section is equivalent to pass the mixed-phase sequence through an proper allpass filter to become a minimum phase output sequence. Intuitively, this can be inferred from the fact that the mixed-phase sequence and its minimum-phase counterpart have the same magnitude response with different phase response. Formally, We can prove this by considering the characteristics of an allpass filter's complex cepstrum.

An allpass filter's transfer function $H_{ap}(z)$ can be expressed as

$$H_{ap}(z)=\frac{\prod_{k=1}^{N}(1-a_k z^{-1})}{\prod_{k=1}^{N}(1-a_k z)}Z^{N} \quad where \quad |a_k|<1 \tag{11}$$

We can drop the linear-phase term z^N to compute its complex cepstrum as follow:

$$\begin{aligned}&\log\left[H_{ap}(z)\right]\\&=-\sum_{k=1}^{N}\log(1-a_k z)+\sum_{k=1}^{N}\log(1-a_k z^{-1})\\&=\sum_{k=1}^{N}\sum_{n=1}^{\infty}\frac{a_k^n}{n}z^n-\sum_{k=1}^{N}\sum_{n=1}^{\infty}\frac{a_k^n}{n}z^{-n}\\&=\sum_{n=-\infty}^{-1}\left(\sum_{k=1}^{N}\frac{-a_k^{-n}}{n}\right)z^{-n}+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N}\frac{-a_k^{n}}{n}\right)z^{-n}\end{aligned} \tag{12}$$

Thus,

$$\hat{h}_{ap}(n)=\begin{cases}\sum_{k=1}^{N}\frac{-a_k^n}{n} & n>0\\ 0 & n=0\\ \sum_{k=1}^{N}\frac{-a_k^{-n}}{n} & n<0\end{cases} \tag{13}$$

Notice that an allpass filter's complex cepstrum is an odd sequence. From the previous discussion, while we drop the non-causal part of the mixed-phase sequence, it can be viewed as to apply an allpass filter whose non-causal part is just the negative non-causal part of the mixed phase sequence as shown in Fig. 1(c). Therefore, the two non-causal parts will be added and cancelled to be zero. Moreover, if we consider the zeros' locations, we find that, by multiplying $H(z)$ with $H_{ap}(z)$, the effect is that the $H(z)$'s zeros lying outside the unit circle will be cancelled by the poles of $H_{ap}(z)$'s denominator, and reflected inside unit circle at their reciprocal conjugate locations.

5. DESIGN EXAMPLES

In the following, two examples are given to illustrate the design of minimum-phase FIR filter by the proposed method. Among the two examples, the first one is a linear-phase equiripple lowpass filter. The other one is a lowpass mixed-phase filter which emphasizes that we need not to start with a linear-phase filter to accomplish our work. In the two cases, their original frequency magnitude/phase response and zero-pole plot are shown in Fig.2 and Fig.3, respectively.

6. CONCLUSION

We have introduced a simple effective method to construct a minimum-phase FIR filter from a mixed-phase filter. In Table 1, we compare the new method with other ones proposed in some literature. In our design procedure and the procedure in [6], the efforts required are merely two FFTs and a recursive procedure to compute the impulse response either from real cepstrum or from root moments. Therefore the complexities are lowest. Note that coefficients scaling are used to handle the unit-circle zeros' numerical problem. Also, while we compute the real cepstrum sequence, zero-padding are necessary to reduce the aliasing effect. During the process, there's no need to unwrap the phase or find the roots. Neither do we need to begin with an odd-length equiripple linear-phase filter and get the square root magnitude response. The resultant minimum phase filter magnitude response will be exactly the same as the original one.

Table 1. Comparison between the proposed method and several existing methods in the open literature.

	[1]	[2]	[3]	[6]	[4]	proposed method
Herrmann's approach [1]	v	v	v	v*1	v	
factorization*2	RF	CC	DC	RM	NR	RC
Phase-unwrapping		v				
FFT		2	3	2		2
Iteration	v*3				v	
minimum-phase filter's magnitude	square root*4	square root	square root	the same	square root	the same
Low complexity				v		v

*1: The prototype is not necessarily an equiripple one.

*2: RF: root-finding, CC: complex cepstrum, DC: differential cepstrum, RM: root moment, NR: Newton-Raphson iteration, RC: real cepstrum

*3: The specific iterative procedure depends on which root-finding algorithm is applied.

*4: Square root of the prototype's magnitude response.

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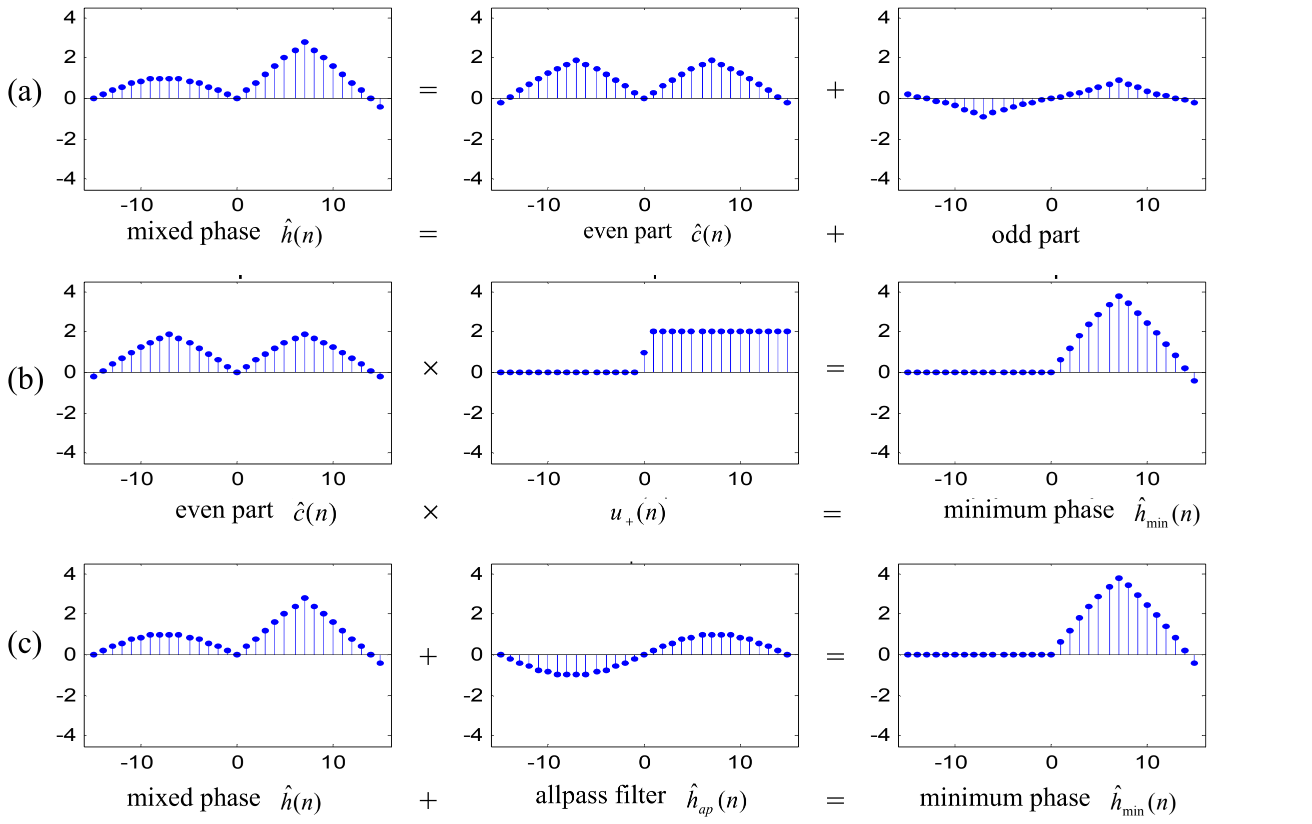

Fig. 1. Construction of minimum phase filter from a mixed phase sequence. (a) Real cepstrum $\hat{c}(n)$, i.e. the even part of complex cepstrum $\hat{h}(n)$, can be effectively computed by $IFFT_L\{\log|H(e^{j\omega})|\}$ without phase unwrapping and root-finding. (b) Reconstruct $\hat{h}_{\min}(n)$ from its even part $\hat{c}(n)$. (c) $\hat{h}_{\min}(n)$ is obtained by passing $\hat{h}(n)$ through a proper allpass filter $\hat{h}_{ap}(n)$ by canceling its non-causal part.

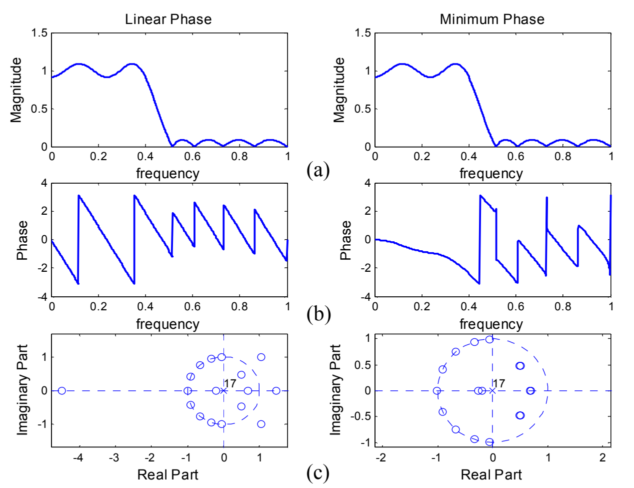

Fig. 2. Equiripple lowpass linear phase and minimun phase filters with length 18 (a) Amplitude response (b) Phase response and (c) Zero-pole plot.

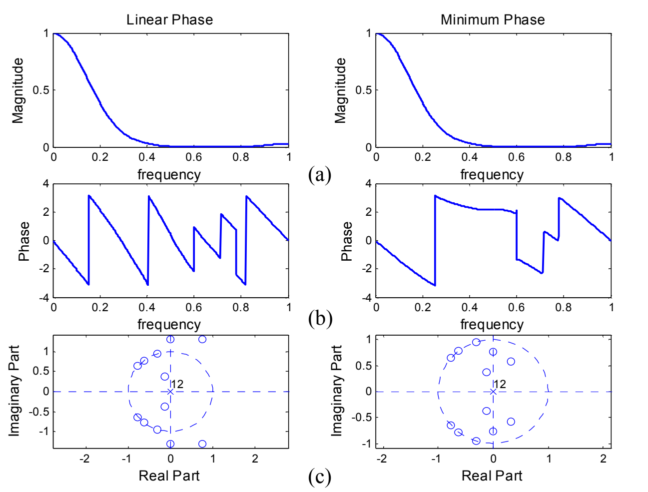

Fig. 3. Lowpass mixed phase and minimum phase filters with length 13 (a) Amplitude response (b) Phase response and (c) Zero-pole plot.