LINE SPECTRAL PROPERTIES OF QUADRATIC MODELS

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ABSTRACT

Line Spectrum Pair (LSP) decomposition is a technique developed for robust representation of the coefficients of a Linear Predictive (LP) model. It has favourable properties with respect to root loci and quantisation noise. In this article, we will explore the properties of LSP polynomials when they are used to represent quadratic models of form $A^2(z)$ and $A(z)z^{-m}A(z^{-1})$. The quadratic models show intriguing properties in LSP decomposition, which can be used to develop a Levinson-type algorithm.

1. INTRODUCTION

In speech coding, th Line Spectrum Pair (LSP) decomposition is a widely used representation of coefficients of the Linear Predictive (LP) model [1]. The mathematical properties were first presented by Schüssler [2], even though it was Soong who brought them into the attention of speech coding community [3]. Their use in speech coding is warranted by several favourable properties; i) the stability of the synthesis filter is easily translated to interlacing properties of Line Spectral Frequencies (LSFs), ii) the representation is robust in the presence of quantisation noise and iii) the LSF domain is, in speech coding, a perceptually suitable domain for inter-frame interpolation [4]. Due to these reasons, LSP polynomials are the most widely used representation for LP coefficients and, moreover, they are also well appreciated as features in speech recognition.

However, the mathematical properties of the LSP decomposition have generally not been as much appreciated. For one thing, the LSP polynomials have a close connection to the Levinson-recursion, which can be used to solve equations with Toeplitz matrices [5]. In addition, the LSFs have interesting inter-model interlacing properties [6]. In this article, we will show yet more intriguing mathematical properties of the LSP decomposition in relation to quadratic models.

2. BACKGROUND

2.1 Line Spectrum Pair

In contrast to convention of Soong [3], we will adopt a slightly more general notation to accommodate for the more general decomposition defined by Schüssler [2]. Thus, we will define the LSP decomposition of an mth order polynomial $A(z)$ as

$$\begin{aligned} \mathcal{P}_k[A(z)] &= A(z) + z^{-m-k}A(z^{-1}) \\ \mathcal{Q}_k[A(z)] &= A(z) - z^{-m-k}A(z^{-1}) \end{aligned} \tag{1}$$

where $k \geq 0$ is an integer. For $k = 1$ we obtain the conventional form of LSP. It follows trivially for all $k \geq 0$ that

$$A(z) = \frac{1}{2}\left\{\mathcal{P}_k[A(z)] + \mathcal{Q}_k[A(z)]\right\}. \tag{2}$$

In the following, when there is no danger of confusion, we will omit $A(z)$ from $\mathcal{P}_k[A(z)]$ and $\mathcal{Q}_k[A(z)]$ and write simply $\mathcal{P}_k(z)$ and $\mathcal{Q}_k(z)$.

Provided that $A(z)$ has its zeros inside the unit circle, the LSP polynomials $\mathcal{P}_k(z)$ and $\mathcal{Q}_k(z)$ are, respectively, symmetric and antisymmetric, that is, $\mathcal{P}_k(z) = z^{-m-k}\mathcal{P}_k(z^{-1})$ and $\mathcal{Q}_k(z) = -z^{-m-k}\mathcal{Q}_k(z^{-1})$. Moreover, their zeros are interlaced on the unit circle. Conversely, these properties are necessary and sufficient to guarantee that $A(z)$ has its zeros inside the unit circle [2, 3].

2.2 Levinson recursion

In the solution of symmetric Toeplitz equations, the Levinson recursion is a classic [5, 7]. On the mth iteration we have [7]

$$\mathbf{R}_m\mathbf{a}_m = [\ {}^2_m 0\,0 \dots 0]^T \tag{3}$$

where $\mathbf{R}_m$ is the $m \times m$ principal sub-matrix of a symmetric Toeplitz matrix $\mathbf{R}$, vector $\mathbf{a}_m$ is the $m \times 1$ solution to the Toeplitz problem and ${}_m$ a scalar.

On each recursion step we define

$$\mathbf{a}_{m+1}^T = [\mathbf{a}_m^T, 0] + {}_m[0, \mathbf{J}\mathbf{a}_m^T] \tag{4}$$

where $\mathbf{J}$ is the row reversal matrix and ${}_m$ is the mth reflection coefficient. This coefficient ${}_m$ is defined such that the residual energy ${}^2_{m+1} = \mathbf{a}_{m+1}^T\mathbf{R}_{m+1}\mathbf{a}_{m+1}$ is minimised, which also ensures that the structure of Eq. 3 is retained on the $(m+1)$th step. The relation of Eq. 4 can be expressed in terms of the LSP polynomials as

$$A_{m+1}(z, {}_m) = \frac{1 + {}_m}{2}\mathcal{P}_1[A_m(z)] + \frac{1 - {}_m}{2}\mathcal{Q}_1[A_m(z)]. \tag{5}$$

Often, when the explicit value of ${}_m$ is irrelevant, we will omit ${}_m$ from $A_{m+1}(z, {}_m)$ and write $A_{m+1}(z)$.

Note that the zeros of $A_{m+1}(z, {}_m)$ are symmetric with ${}_m$ and ${}^{-1}_m$ with respect to the unit circle. That is,

$$A_{m+1}(z, {}_m) = \ z^{-m-1}A_{m+1}(z^{-1}, {}^{-1}_m) \tag{6}$$

where is a scaling coefficient [8]. The properties of $A_{m+1}(z, {}_m)$ and $A_{m+1}(z, {}^{-1}_m)$ are thus, on the unit circle, equivalent and moreover, ${}_m$ and ${}^{-1}_m$ are in this sense interchangeable. However, in order to retain stability, we have to choose such a ${}_m$ that all poles of $A^{-1}_{m+1}(z, {}_m)$ are within the unit circle.

3. QUADRATIC FORMS

With *quadratic forms* of order $2m$ we refer to two kinds of polynomials, the direct and indirect forms, which we will define as $A^2(z)$ and $A(z)z^{-m}A(z^{-1})$, respectively. The two forms are related by the property

$$\left|A^2(z)\right| = \left|A(z)z^{-m}A(z^{-1})\right| \text{ for } z = e^{i} \ . \tag{7}$$

It is immediately obvious that the indirect form $A(z)z^{-m}A(z^{-m})$ is symmetric, whereas the structure of $A^2(z)$ is hidden in the coefficients. However, the direct quadratic form is more practical in applications, since its inverse can be made stable, but we shall see in the following that the indirect form has other, readily tractable, unsurpassed properties.

3.1 Residual energy

The residual energy $\ ^2$ of a linear predictive model is given by $\ ^2 = \mathbf{a}^T\mathbf{R}\mathbf{a}$, where matrix $\mathbf{R}$ is the symmetric Toeplitz autocorrelation matrix of the input signal and $\mathbf{a}$ is the coefficient vector of the model.

For a quadratic model the residual energy can be calculated as follows. Let $\mathbf{a}$ and $\mathbf{C_a}$ be the coefficient vector and convolution matrix of $A(z)$. The direct quadratic form $A^2(z)$ is then, in matrix form, $\mathbf{C_a a}$. The indirect form $A(z)z^{-m}A(z^{-1})$ is $\mathbf{C_a J a}$, where matrix $\mathbf{J}$ is the row-reversal matrix.

For the residual of the indirect and direct quadratic models $\ _i$ and $\ _d$, respectively, we then have

$$\ _i^2 = \mathbf{a}^T\mathbf{J}^T\mathbf{C_a}^T\mathbf{R}\mathbf{C_a}\mathbf{J}\mathbf{a} = \mathbf{a}^T\mathbf{C_a}^T\mathbf{R}\mathbf{C_a}\mathbf{a} = \ _d^2 \tag{8}$$

since $\mathbf{J}^T\mathbf{C_a}^T\mathbf{R}\mathbf{C_a}\mathbf{J} = \mathbf{C_a}^T\mathbf{R}\mathbf{C_a}$ by symmetry. In other words, the residual of the indirect and direct quadratic forms are equal. Consequently, minimisation of the residual energy of the indirect form also minimises the direct form.

3.2 Quadratic Line Spectrum Pair

First of all, note that the squared LSP polynomials have the following trivial properties

$$\begin{aligned} \mathcal{P}_k^2(z) &= \ \ \mathcal{P}_k(z)z^{-m-k}\mathcal{P}_k(z^{-1}) \\ \mathcal{Q}_k^2(z) &= -\mathcal{Q}_k(z)z^{-m-k}\mathcal{Q}_k(z^{-1}) \end{aligned} \tag{9}$$

and by substitution of Eq. 2 we obtain

$$\begin{aligned} \mathcal{P}_k^2(z) &= A^2(z) + z^{-2m-2k}A^2(z^{-1}) + 2A(z)z^{-m-k}A(z^{-1}) \\ \mathcal{Q}_k^2(z) &= A^2(z) + z^{-2m-2k}A^2(z^{-1}) - 2A(z)z^{-m-k}A(z^{-1}). \end{aligned} \tag{10}$$

With these relationships, we can rewrite the LSP decomposition of the direct quadratic form as

$$\begin{aligned} \mathcal{P}_k[A^2(z)] &= \frac{1}{4}\left(1+z^{2l-k}\right)\left\{\mathcal{P}_l^2[A(z)] + \mathcal{Q}_l^2[A(z)]\right\} \\ &\quad + \frac{1}{2}\left(1-z^{2l-k}\right)\mathcal{P}_l[A(z)]\mathcal{Q}_l[A(z)] \\ \mathcal{Q}_k[A^2(z)] &= \frac{1}{4}\left(1-z^{2l-k}\right)\left\{\mathcal{P}_l^2[A(z)] + \mathcal{Q}_l^2[A(z)]\right\} \\ &\quad + \frac{1}{2}\left(1+z^{2l-k}\right)\mathcal{P}_l[A(z)]\mathcal{Q}_l[A(z)] \end{aligned} \tag{11}$$

where l is an integer. In the trivial case of $2k = l$, these equations reduce to

$$\begin{aligned} \mathcal{P}_{2l}[A^2(z)] &= \frac{1}{2}\left[\mathcal{P}_l^2[A(z)] + \mathcal{Q}_l^2[A(z)]\right] \\ \mathcal{Q}_{2l}[A^2(z)] &= \mathcal{P}_l[A(z)]\mathcal{Q}_l[A(z)]. \end{aligned} \tag{12}$$

Similarly, the indirect quadratic form $A(z)z^{-m}A(z^{-1})$ has

$$\begin{aligned} \mathcal{P}_k\left[A(z)z^{-m}A(z^{-1})\right] &= \frac{1}{4}\left\{\mathcal{P}_l^2[A(z)] - \mathcal{Q}_l^2[A(z)]\right\}\left(1+z^{-k}\right) \\ \mathcal{Q}_k\left[A(z)z^{-m}A(z^{-1})\right] &= \frac{1}{4}\left\{\mathcal{P}_l^2[A(z)] - \mathcal{Q}_l^2[A(z)]\right\}\left(1-z^{-k}\right) \end{aligned} \tag{13}$$

and with $k = 0$ we have ($l \geq 0$)

$$\begin{aligned} \mathcal{P}_0\left[A(z)z^{-m}A(z^{-1})\right] &= \frac{1}{2}\left\{\mathcal{P}_l^2[A(z)] - \mathcal{Q}_l^2[A(z)]\right\} \\ \mathcal{Q}_0\left[A(z)z^{-m}A(z^{-1})\right] &\equiv 0. \end{aligned} \tag{14}$$

With the LSP polynomials defined, we can now continue to study their properties in the next sections.

3.3 Root loci

Given a minimum-phase polynomial $A(z)$, we know that the LSP polynomials (Eq. 1) have their zeros interlaced on the unit circle. Since $A(z)$ is minimum-phase, then also $A^2(z)$ is minimum-phase and its LSP polynomials have also zeros interlaced on the unit circle. From Eq. 12 we can therefore readily conclude that the zeros of $\mathcal{P}_l^2[A(z)] + \mathcal{Q}_l^2[A(z)]$ and $\mathcal{P}_l^2[A(z)]\mathcal{Q}_l^2[A(z)]$ are interlaced. That is, importantly, the roots of $\mathcal{P}_l^2[A(z)] + \mathcal{Q}_l^2[A(z)]$ lie on the unit circle. This rationale holds for l even, but can readily be extended to l odd.

On the other hand, since $\mathcal{P}_0[A(z)z^{-m}A(z^{-1})] = A(z)z^{-m}A(z^{-1})$ and from Eq. 14 we can readily see that $\mathcal{P}_l^2[A(z)] - \mathcal{Q}_l^2[A(z)]$ has all its zeros *off* the unit circle. The root loci of Eq. 13 is less attractive, since it has the same zeros as in Eq. 14, but also trivial roots interlaced on the unit circle.

3.4 Spectral Magnitude and Envelope

In the above section we showed that $\mathcal{P}_k^2(z) - \mathcal{Q}_k^2(z)$ has all its zeros *off* the unit circle, while $\mathcal{P}_k^2(z) + \mathcal{Q}_k^2(z)$ has all zeros *on* the unit circle. However, now we will show that the spectral envelop of $\mathcal{P}_k^2(z) + \mathcal{Q}_k^2(z)$ is equal to the spectrum of $\mathcal{P}_k^2(z) - \mathcal{Q}_k^2(z)$. Recall that the zeros of $\mathcal{P}_k^2(z) + \mathcal{Q}_k^2(z)$ are interlaced with the zeros of $\mathcal{P}_k^2(z)$ and $\mathcal{Q}_k^2(z)$ and that both $\mathcal{P}_k^2(z)$ and $\mathcal{Q}_k^2(z)$ are real and non-negative and non-positive, respectively, on the unit circle (see Appendix A, Lemma 1). Then if the magnitudes of $\mathcal{P}_k^2(z) + \mathcal{Q}_k^2(z)$ and $\mathcal{P}_k^2(z) - \mathcal{Q}_k^2(z)$ are equal, it follows that

i. $\mathcal{P}_k^2(z_i) + \mathcal{Q}_k^2(z_i) = \mathcal{P}_k^2(z_i) - \mathcal{Q}_k^2(z_i)$
$\quad \Rightarrow \quad \mathcal{Q}_k^2(z_i) = -\mathcal{Q}_k^2(z_i) \quad \Rightarrow \quad \mathcal{Q}_k^2(z_i) = 0$

ii. $\mathcal{P}_k^2(z_i) + \mathcal{Q}_k^2(z_i) = -\left[\mathcal{P}_k^2(z_i) - \mathcal{Q}_k^2(z_i)\right]$
$\quad \Rightarrow \quad \mathcal{P}_k^2(z_i) = -\mathcal{P}_k^2(z_i) \quad \Rightarrow \quad \mathcal{P}_k^2(z_i) = 0.$

The magnitudes of $\mathcal{P}_k^2(z) + \mathcal{Q}_k^2(z)$ and $\mathcal{P}_k^2(z) - \mathcal{Q}_k^2(z)$ thus coincide exactly at the discrete zeros of $\mathcal{P}_k(z)$ and $\mathcal{Q}_k(z)$. In other words, $\mathcal{P}_k^2(z) - \mathcal{Q}_k^2(z)$ is the spectral envelope of $\mathcal{P}_k^2(z) + \mathcal{Q}_k^2(z)$. See Fig. 1 for illustration.

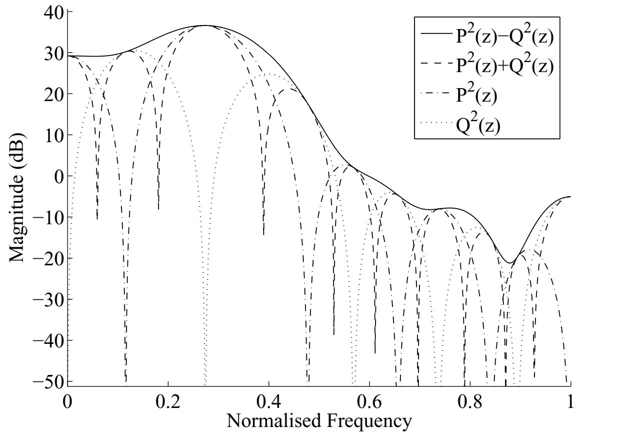

Figure 1: Magnitude spectra of $\mathcal{P}_k^2(z) - \mathcal{Q}_k^2(z)$, $\mathcal{P}_k^2(z) + \mathcal{Q}_k^2(z)$, $\mathcal{P}_k^2(z)$ and $\mathcal{Q}_k^2(z)$.

4. SYMMETRIC LSP METHOD – THE QUADRATIC LEVINSON ALGORITHM

In terms of the Levinson recursion, we have, corresponding to Eqs. 2 and 5, with $P_m(z) = \mathcal{P}_1[A_m(z)]$ and $Q_m(z) = \mathcal{Q}_1[A_m(z)]$, the following relation

$$\begin{aligned} A_{m+1}(z)z^{-m-1}A_{m+1}(z^{-1}) &= [\ \ ^2P_m^2(z) - (1-\ \ )^2 Q_m^2(z)] \\ &= \ \ [\ \ P_m^2(z) - (1-\ \ )Q_m^2(z)] \end{aligned} \quad (15)$$

where $= \frac{1+}{2}$, and $= \frac{-\sqrt{\ (1-\ )}}{2\ -1}$ are scalars. Similar equations for the direct quadratic form can be calculated, but since they contain cross-terms of $P(z)$ and $Q(z)$ they are not as tractable.

Since the spectral properties of $P_m^2(z) + Q_m^2(z)$ and $P_m^2(z) - Q_m^2(z)$ are similar, we can, instead of Eq. 15, on each iteration optimise the modified quadratic function

$$(\ \ ) = \ \ P_m^2(z) + (1-\ \ )Q_m^2(z). \quad (16)$$

In the following, we will find that this modified function has tractable properties in view of the residual energy optimisation.

We are now ready to develop a Levinson-type recursion for the quadratic form. The initial condition is $A_0(z) = 1$. Our objective is, for each iteration m, to minimise the residual energy with the quadratic form defined as in Eq. 16.

Denote the coefficient vectors of $P_m^2(z)$ and $Q_m^2(z)$ by $\tilde{\mathbf{p}}_m$ and $\tilde{\mathbf{q}}_m$, respectively. The residual energy $\hat{\ }_m^2$ of the modified quadratic function $(\ \ )$ (Eq. 16) is then

$$\hat{\ }_m^2 = [\ \ \tilde{\mathbf{p}}_m^T + (1-\ \ )\tilde{\mathbf{q}}_m^T]\,\mathbf{R}_m\,[\ \ \tilde{\mathbf{p}}_m + (1-\ \ )\tilde{\mathbf{q}}_m]. \quad (17)$$

The minimal residual energy can be calculated by partial differentiation and setting to zero $\hat{\ }_m^2/\ \ = 0$. Solving for yields

$$= \frac{(\tilde{\mathbf{q}}_m - \tilde{\mathbf{p}}_m)^T \mathbf{R}_m \tilde{\mathbf{q}}_m}{(\tilde{\mathbf{p}}_m - \tilde{\mathbf{q}}_m)^T \mathbf{R}_m (\tilde{\mathbf{p}}_m - \tilde{\mathbf{q}}_m)}. \quad (18)$$

We can readily see that the denominator is positive since $\mathbf{R}$ is positive definite. Likewise, in the numerator, the term $\tilde{\mathbf{q}}^T\mathbf{R}\tilde{\mathbf{q}}$ is positive, while the term $\tilde{\mathbf{p}}^T\mathbf{R}\tilde{\mathbf{q}}$ is shown to be negative in Appendix A, Lemma 2. Consequently, is always positive > 0.

In Eq. 15, we could have, without loss of generality, defined $= 1-$ and arrived to an equation for not much unlike Eq. 18. By the same rationale we find that > 0 and thus $= 1- \ > 0$. Combining, we have $\in (0,1)$.

As noted earlier, we can use this optimal from Eq. 18 in Eq. 15 albeit we used Eq. 16 in optimisation.

4.1 Algorithm

The Levinson-type algorithm for quadratic models can then be stated as

1. Let $A_0(z) = 1$ and $m = 0$.
2. Calculate $P(z)$ and $Q(z)$ from Eq. 1.
3. Calculate from Eq. 18 and let $= \frac{-\sqrt{\ (1-\ )}}{2\ -1}$.
4. Set $A_m(z) = \ \ P(z) + (1-\ \ )Q(z)$.
5. If $m \geq N$ then stop. Otherwise let $m := m+1$ and return to Step 2.

The resulting model is the direct quadratic form $A_N^2(z)$.

5. ANTISYMMETRIC LSP METHOD

The Quadratic Levinson algorithm was based on optimising Eq. 16. Comparing to Eq. 12, we notice that in practice, we have only optimised the *symmetric* LSP polynomial of $A^2(z)$. However, the *antisymmetric* LSP polynomial of $A^2(z)$ in Eq. 12 has significant value in itself. Namely, we can, from any antisymmetric polynomial with zeros only on the unit circle, construct a squared model with an identical antisymmetric LSP polynomial.

Specifically, given an antisymmetric polynomial $E(z)$ with zeros on the unit circle, we can find its zeros and organise them in two interlacing sets. Based on these two sets of zeros, construct two polynomials $P(z)$ and $Q(z)$. Letting $\mathcal{P}_1(z) = P(z)$ and $\mathcal{Q}_1(z) = Q(z)$, and by using Eq. 2 we obtain a polynomial $A(z) = \frac{1}{2}[P(z) + Q(z)]$ which fulfills Eq. 12, that is, $\mathcal{Q}_2[A^2(z)] = \ \ E(z)$, where is a scalar.

This does not, however, yet give us an optimal choice of $E(z)$. A well-warranted choice would be the antisymmetric LSP model of higher order such as

$$\mathbf{Rq} = \frac{\ ^2}{2}[+1, 0, 0, \ldots 0, -1]^T \quad (19)$$

where $\mathbf{R}$ is the $N \times N$ symmetric Toeplitz positive definite autocorrelation matrix and $\mathbf{q}$ the coefficient vector of the desired antisymmetric polynomial $E(z)$. Another choice would be the antisymmetric eigenvector corresponding to the smallest eigenvalue of $\mathbf{R}$. The eigenvector corresponding to the smallest eigenvalue is not, however, always antisymmetric and therefore we may not always find a suitable antisymmetric eigenvector. This topic is therefore left for further study.

6. EXPERIMENTS AND DISCUSSION

Figure 2 shows magnitude spectrum performance of the Quadratic Levinson algorithm and the antisymmetric LSP method, compared to original signal and conventional LP. Each model is of the same model order $m = 10$. Informal experiments show that both quadratic models tend to show larger formant bandwiths and magnitudes, as is expected.

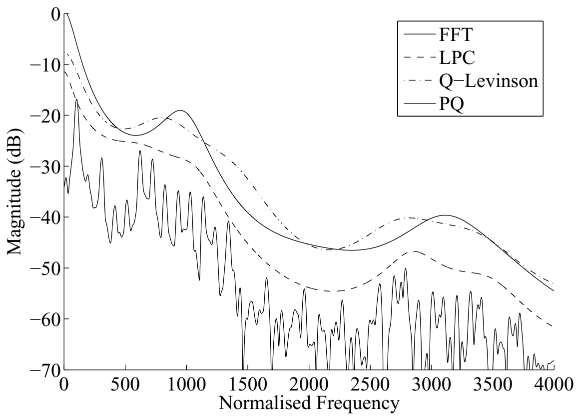

Figure 2: The magnitude spectra of a male vowel /a:/ (FFT), conventional LP model, Quadratic Levinson and the antisymmetric LSP method (PQ). Each spectra is separated by 5dB for visual clarity and each model is of the same order $m = 12$.

The improvements generally come at the expense of formant resolution; closely spaced formants are not as accurately modelled as in conventional LP.

In the Quadratic Levinson algorithm, the final model $A_N^2(z)$ is optimal in the sense that on each iteration the algorithm minimises the residual energy of the modified model with equal spectral envelop. However, minimisation on each recursion step does not guarantee that the overall algorithm reaches the global minimum.

On the other hand, the antisymmetric LSP method is optimised (depending on the method) only for the antisymmetric part of the squared model. Experiments show that the symmetric part, which is determined by the method, yields often a residual which is larger by an order of magnitude.

Concluding, we have presented theory and algorithms for quadratic models using properties of Line Spectrum Pair decomposition. The theoretical results offer intriguing insight into the LSP polynomials, but a lot of work remains in their application.

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A. APPENDIX

Lemma 1. *Let $A(z) = \ _{k=-m}^{m} a_k z^k$ be a symmetric polynomial with all zeros on the unit circle each of multiplicity two and the first (and last) coefficient positive. Then*

1. $A(e^{i\ }) \in \mathbb{R}$ *and* $A(e^{i\ }) \geq 0$ *if* $A(1) \neq 0$.
2. $A(e^{i\ }) \in \mathbb{R}$ *and* $A(e^{i\ }) \leq 0$ *if* $A(1) = 0$.

Proof. Since each zero $\ = e^{i\ }$ of $A(z)$ is of even multiplicity, we can readily see that $\mathrm{sign}[A(e^{i(\ +\ )})] = \mathrm{sign}[A(e^{i(\ -\ )})]$ for a sufficiently small . It follows that the sign of $A(e^{i\ })$ is thus equal for all values .

If $A(z)$ does not have a zero at $z = 1$, then $A(e^{i\ }) \geq 0$ since the value of $A(z)$ is continuous and $A(z) \geq 0$ for $z \in \mathbb{R}$. The first condition is thus satisfied.

If, however, $A(z)$ has a double zero at $z = 1$, then it exhibits a change of sign when going from the real axis to the unit circle. This can be observed by setting two single zeros at $z = e^{\pm i\ }$ and letting approach zero $\ \to 0$. Due to continuity, the second condition is then satisfied. □

Lemma 2. *Let $P^2(z)$ and $Q^2(z)$ be two symmetric polynomials with all zeros on the unit circle each of even multiplicity, and let $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ be the coefficient vectors of $P^2(z)$ and $Q^2(z)$, respectively. Furthermore, let $\mathbf{R}$ be a real, positive definite, symmetric Toeplitz matrix. Then*

1. $\hat{\mathbf{p}}\mathbf{R}\hat{\mathbf{q}} > 0$ *if neither or both $P^2(z)$ and $Q^2(z)$ have a double zero at $z = 1$.*
2. $\hat{\mathbf{p}}\mathbf{R}\hat{\mathbf{q}} < 0$ *if either but not both of $P^2(z)$ and $Q^2(z)$ have a double zero at $z = 1$.*

Proof. From [9, 10] we know that a Toeplitz matrix $\mathbf{R}$ can, with a Vandermonde matrix $\mathbf{V}$ and diagonal matrix $\mathbf{D}$, be decomposed as $\mathbf{R} = \mathbf{V}^T\mathbf{D}\mathbf{V}$. The decomposition is ambiguous and we can thus choose one, or, in the special case of exponentials on the unit circle, two of the Vandermonde rows. This specifies the whole matrix [10]. The matrix $\mathbf{D}$ has positive coefficients iff $\mathbf{R}$ is positive definite.

Recall that an $m \times m$ Vandermonde matrix consist of rows of exponential progressions $\ _k^n$. The product $\mathbf{Va}$ of a vector $\mathbf{a}$ and the Vandermonde matrix $\mathbf{V}$ thus corresponds to evaluating the polynomial $A(z)$ at points $\ _k$.

We can then rewrite the product as $\hat{\mathbf{p}}^T\mathbf{R}\hat{\mathbf{q}} = \hat{\mathbf{p}}^T\mathbf{V}^T\mathbf{D}\mathbf{V}\hat{\mathbf{q}}$. Due to Lemma 1, the values of $P^2(z)$ and $Q^2(z)$ are always positive and negative, respectively, $P^2(e^{i\ }) \geq 0$ and $Q^2(e^{i\ }) \leq 0$ and, consequently, $\mathbf{V}\hat{\mathbf{p}} \geq 0$ and $\mathbf{V}\hat{\mathbf{q}} \leq 0$. Finally, since $\mathbf{D}$ is strictly positive, then $\hat{\mathbf{p}}^T\mathbf{V}^T\mathbf{D}\mathbf{V}\hat{\mathbf{q}} < 0$. □