AN EFFICIENT SVD UPDATE ALGORITHM AND APPLICATIONS TO MIMO COMMUNICATIONS

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ABSTRACT

An efficient update algorithm is presented which tracks the left and right singular vectors and singular values of a transfer matrix, using input and output vectors and without explicitly computing the matrix itself. Such an algorithm has many potential applications in multiple-input multiple-output wireless communication systems in which the channel parameters change slowly with time. Examples of two such applications, eigenbeamforming and adaptive V-BLAST, are presented.

1. INTRODUCTION

There has been vast interest in recent years in the exploitation of spatial diversity through the use of multiple element antennas (MEAs) at each end of a communications link, generally referred to as multiple-input multiple-output (MIMO) systems. In this scenario, the channel response is described by a complex matrix, such that the equivalent channel model is given by the linear system

$$\mathbf{r}_k = \mathbf{H}\mathbf{s}_k + \mathbf{n}_k$$

where the N_t elements of $\mathbf{s}_k$ are the input signals at the transmitter antenna elements, sampled at intervals k, and the corresponding output signals at the N_r receive antennas are the elements of $\mathbf{r}_k$. The additive noise is $\mathbf{n}_k$. The transfer function $\mathbf{H}$ may be static or time-varying – for simplicity, it will be assumed herein to be static.

In many applications, two examples of which are given herein, the signal processing requires the singular value decomposition (SVD) of the $N_r \times N_t$ matrix $\mathbf{H}$, which is given by

$$\mathbf{H} = \mathbf{U} \quad \mathbf{V}^H$$

where the columns of the $N_r \times N_r$ matrix $\mathbf{U} = [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_{N_r}]$ and the $N_t \times N_t$ matrix $\mathbf{V} = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_{N_t}]$ are the left and right singular vectors, respectively. The elements of the $N_r \times N_t$ matrix are non-zero only on the diagonal, with $_{ii} = \ _i$ for $i = 1, \ldots, \min(N_t, N_r)$ where $_i$ is the ith largest singular value. For simplicity of presentation in the following, it will be assumed that $N_t = N_r = N$.

Communications systems which operate on frame-based data typically include a training sequence in each frame from which the channel matrix is estimated. The SVD is then generated from this estimate, requiring significant computation. The main source of performance degradation is the deviation of the estimated singular vectors from their true values, which is highly dependent on the accuracy of the channel estimate.

The channel matrix typically varies slowly with respect to the symbol interval, therefore it might be possible to achieve a better estimate of the SVD components if the channel parameters are 'learned', building on prior knowledge to improve the estimate at each step [1]. Furthermore, the complexity may be reduced if the SVD components, i.e., the singular vectors and singular values, were estimated directly without explicitly computing a channel matrix estimate.

This paper presents an algorithm which updates the estimates of some or all of the singular values and the corresponding left and right singular vectors, based on prior estimates. In Sec. 3, two applications are presented in which the algorithm can be used to avoid the explicit estimation of $\mathbf{H}$.

The underlying update equation is

$$\mathbf{H}_n = \quad \mathbf{H}_{n-1} + (1 - \quad)\mathbf{r}_n\mathbf{s}_n^H \tag{1}$$

where $n = kN_p$, i.e., the updates are based on known training symbol vectors inserted periodically into the unknown data stream $\mathbf{s}_k$. Thus, the objective is to determine the SVD components which satisfy the relationship

$$\mathbf{U}_n \quad _n\mathbf{V}_n^H = \quad \mathbf{U}_{n-1} \quad _{n-1}\mathbf{V}_{n-1}^H + (1 - \quad)\mathbf{r}_n\mathbf{s}_n^H.$$

A full SVD computation has a complexity $O(N^3)$. Subspace tracking techniques have been proposed which track the eigenvalues $_{n,i} = \ ^2_{n,i}$ and eigenvectors (left singular vectors) of the covariance matrix $\mathbf{R} = \mathscr{E}\left\{\mathbf{r}_k\mathbf{r}_k^H\right\}$, for example in [2]. In [3], a method was presented for updating the eigenvalues and right singular vectors; in both these cases, obtaining the other set of singular vectors requires computationally expensive matrix multiplications. An adaptive algorithm was proposed in [4] for the MIMO application described in Sec. 3.1, in which a small perturbation was applied at the transmitter (right singular vectors), and the preferred sign for its application to the transmitter weight was determined at the receiver and fed back. This method has a complexity of $O(N^3)$ and converges more slowly and with less accuracy than the algorithm proposed herein, which has complexity $O(N^2)$. The new algorithm has a similar form to that presented in [2], but addresses the more complex problem of determining the singular values and both left and right singular vectors and is derived in a more concise fashion.

2. SVD UPDATE ALGORITHM

The updated transfer function can be written as a first order perturbation of the prior estimate, i.e.,

$$\mathbf{H}_n = \mathbf{H}_{n-1} + \mathbf{E} \tag{2}$$

where, from (1),

$$\mathbf{E} = (1 - \quad)\left(\mathbf{r}_n\mathbf{s}_n^H - \mathbf{H}_{n-1}\right). \tag{3}$$

The matrices of singular vectors at update n can be written as rotations of those at update $n-1$ as follows

$$\mathbf{U}_n = \mathbf{U}_{n-1}(\mathbf{I}+\mathbf{A}) \quad \text{and} \quad \mathbf{V}_n = \mathbf{V}_{n-1}(\mathbf{I}+\mathbf{B}). \quad (4)$$

As $\mathbf{U}_{n-1}$ and $\mathbf{V}_{n-1}$ are unitary, up to a first order approximation,

$$\mathbf{A}+\mathbf{A}^H = \mathbf{B}+\mathbf{B}^H = 0. \quad (5)$$

The perturbation equation (2) can be rewritten to give

$$\mathbf{H}_n = \mathbf{U}_{n-1}(\ \ _{n-1}+\mathbf{F})\,\mathbf{V}_{n-1}^H.$$

where $\mathbf{F} = \mathbf{U}_{n-1}^H\mathbf{E}\mathbf{V}_{n-1}$, i.e., from (3)

$$\mathbf{F} = (1-\ \ )\left(\mathbf{U}_{n-1}^H\mathbf{r}_n\mathbf{s}_n^H\mathbf{V}_{n-1} - \ \ _{n-1}\right). \quad (6)$$

Denoting the ith columns of $\mathbf{U}_{n-1}$ and $\mathbf{V}_{n-1}$ by $\mathbf{u}_{n-1,i}$ and $\mathbf{v}_{n-1,i}$, respectively, the (i,j)th element of $\mathbf{F}$ is given by

$$f_{ij} = \mathbf{u}_{n-1,i}^H\mathbf{E}\mathbf{v}_{n-1,j}.$$

2.1 Singular value update

It is shown in [5, Sec. V.4.2] that the singular values of the perturbed matrix, $\mathbf{H}_n$, are approximated by

$$\ \ _{n,i} = \ \ _{n-1,i} + f_{ii} + O(\|\mathbf{E}\|_2^2).$$

Using the definition (6) and noting that the singular values are real, the singular value update equation is

$$\ \ _{n,i} = \ \ \ \ _{n-1,i} + (1-\ \ )\mathscr{R}\left\{\mathbf{u}_{n-1,i}^H\mathbf{r}_n\mathbf{s}_n^H\mathbf{v}_{n-1,i}\right\}.$$

2.2 Singular vector update

From (4), the singular vector update equations are

$$\mathbf{u}_{n,i} = \mathbf{u}_{n-1,i} + {}_{j=1}^{N}\ a_{ji}\mathbf{u}_{n-1,j} \qquad i=1,\ldots,N \quad (7)$$

$$\mathbf{v}_{n,i} = \mathbf{v}_{n-1,i} + {}_{j=1}^{N}\ b_{ji}\mathbf{v}_{n-1,j} \qquad i=1,\ldots,N. \quad (8)$$

where the (i,j)th elements of $\mathbf{A}$ and $\mathbf{B}$ are $a_{ij} = \mathbf{u}_{n-1,i}^H\mathbf{u}_{n,j}$ and $b_{ij} = \mathbf{v}_{n-1,i}^H\mathbf{v}_{n,j}$, respectively, for $i \neq j$, and $a_{ii} = b_{ii} = 0$. The result in [5, Sec. V.4.2] can be used to show that for $i<j$

$$a_{ji} = \frac{\left(\ \ _{n-1,i}f_{ji} + \ \ _{n-1,j}f_{ij}^*\right)}{(\ \ _{n-1,i}^2 - \ \ _{n-1,j}^2)} + O(\|\mathbf{E}\|_2^2)$$

$$b_{ji} = \frac{\left(\ \ _{n-1,i}f_{ij}^* + \ \ _{n-1,j}f_{ji}\right)}{(\ \ _{n-1,i}^2 - \ \ _{n-1,j}^2)} + O(\|\mathbf{E}\|_2^2).$$

Also, from (5), $a_{ij} = -a_{ji}^*$ and $b_{ij} = -b_{ji}^*$.

The singular values are ordered such that $\ _i > \ _j$ for $i<j$. The elements of the rotation matrices can therefore be approximated using

$$a_{ji} \approx \frac{f_{ji}}{\ \ _{n-1,i}} = \frac{(1-\ \ )\mathbf{u}_{n-1,j}^H\mathbf{r}_n\mathbf{s}_n^H\mathbf{v}_{n-1,i}}{\ \ _{n-1,i}} \qquad i<j$$

and

$$b_{ji} \approx \frac{f_{ij}^*}{\ \ _{n-1,i}} = \frac{(1-\ \ )\mathbf{v}_{n-1,j}^H\mathbf{s}_n\mathbf{r}_n^H\mathbf{u}_{n-1,i}}{\ \ _{n-1,i}} \qquad i<j.$$

Writing $x_i = \sqrt{1-\ \ }\ \mathbf{v}_{n-1,i}^H\mathbf{s}_n$ and $y_i = \sqrt{1-\ \ }\ \mathbf{u}_{n-1,i}^H\mathbf{r}_n$, the off-diagonal elements of $\mathbf{F}$ are

$$f_{ij} = x_j^* \cdot y_i \qquad i \neq j.$$

The update equations (7) and (8) then become, for $i = 1,\ldots,N$,

$$\mathbf{u}_{n,i} = \mathbf{u}_{n-1,i} + {}_{j=i+1}^{N}\ \frac{x_i^*y_j}{\ \ _{n-1,i}}\mathbf{u}_{n-1,j} - {}_{j=1}^{i-1}\ \frac{x_jy_i^*}{\ \ _{n-1,j}}\mathbf{u}_{n-1,j}$$

$$\mathbf{v}_{n,i} = \mathbf{v}_{n-1,i} + {}_{j=i+1}^{N}\ \frac{x_jy_i^*}{\ \ _{n-1,i}}\mathbf{v}_{n-1,j} - {}_{j=1}^{i-1}\ \frac{x_i^*y_j}{\ \ _{n-1,j}}\mathbf{v}_{n-1,j}.$$

These can be efficiently updated by noting the following relationships

$$\mathbf{p}_0^u = {}_{j=1}^{N}\ y_j\mathbf{u}_{n-1,j} = \sqrt{1-\ \ }\ \mathbf{r}_n$$

$$\mathbf{p}_i^u = {}_{j=i+1}^{N}\ y_j\mathbf{u}_{n-1,j} = \mathbf{p}_{i-1}^u - y_i\mathbf{u}_{n-1,i} \qquad i=1,\ldots,N-1$$

and

$$\mathbf{q}_1^u = \mathbf{0}$$

$$\mathbf{q}_{i+1}^u = {}_{j=1}^{i}\ \frac{x_i^*y_j}{\ \ _{n-1,j}}\mathbf{v}_{n-1,j} = \mathbf{q}_i^u + \frac{x_i}{\ \ _{n-1,i}}\mathbf{u}_{n-1,i} \qquad i=1,\ldots,N-1$$

and vectors $\mathbf{p}_i^v$ and $\mathbf{q}_i^v$ have equivalent definitions for the right singular vectors. Then the singular vectors can be efficiently updated by recursing the following equations

$$\mathbf{u}_{n,i} = \mathbf{u}_{n-1,i} + \frac{x_i^*}{\ \ _{n-1,i}}\mathbf{p}_i^u - y_i^*\mathbf{q}_i^u$$

$$\mathbf{v}_{n,i} = \mathbf{v}_{n-1,i} + \frac{y_i^*}{\ \ _{n-1,i}}\mathbf{p}_i^v - x_i^*\mathbf{q}_i^v.$$

After updating, the singular vectors must be normalised to ensure they have unit length.

2.3 Algorithm

The efficient, recursively implemented SVD update algorithm [6] can be summarised as shown in Table 1.

3. APPLICATIONS

The algorithm derived in Sec. 2 has been applied to two MIMO systems which are briefly described here.

3.1 Eigenbeamforming

A fixed wireless MIMO-OFDM broadband access system was considered in [7]. $N=8$ elements were used in the transmitter and receiver arrays, which were mounted approximately 60 m above street level in Toronto, Canada, with the

Table 1: Efficient SVD update algorithm, for p singular values and singular vector pairs.

INITIALISE:

$\mathbf{r}_n = \sqrt{(1-\quad)}\mathbf{r}_n$

$\mathbf{s}_n = \sqrt{(1-\quad)}\mathbf{s}_n$

$\mathbf{q}^u = \mathbf{q}^v = \mathbf{0}$

$\mathbf{p}^u = \mathbf{r}_n$

$\mathbf{p}^v = \mathbf{s}_n$

$i = 1$

RECURSE:

$x_i = \mathbf{v}_{n-1,i}^H \mathbf{s}_n$

$y_i = \mathbf{u}_{n-1,i}^H \mathbf{r}_n$

$\quad = x_i / \quad_{n-1,i}$

$\quad = y_i / \quad_{n-1,i}$

$\mathbf{p}^u = \mathbf{p}^u - y_i \cdot \mathbf{u}_{n-1,i}$

$\mathbf{p}^v = \mathbf{p}^v - x_i \cdot \mathbf{v}_{n-1,i}$

$\mathbf{u}_{n,i} = \mathbf{u}_{n-1,i} + \quad {}^* \mathbf{p}^u - y_i^* \mathbf{q}^u$

$\mathbf{v}_{n,i} = \mathbf{v}_{n-1,i} + \quad {}^* \mathbf{p}^v - x_i^* \mathbf{q}^v$

$\quad_{n,i} = \quad_{n-1,i} + \mathscr{R}\{x_i^* \cdot y_i\}$

$\mathbf{q}^u = \mathbf{q}^u + \quad \mathbf{u}_{n-1,i}$

$\mathbf{q}^v = \mathbf{q}^v + \quad \mathbf{v}_{n-1,i}$

$\mathbf{u}_{n,i} = \mathbf{u}_{n,i} / \|\mathbf{u}_{n,i}\|$

$\mathbf{v}_{n,i} = \mathbf{v}_{n,i} / \|\mathbf{v}_{n,i}\|$

$i = i + 1$

line-of-sight blocked by a small cluster of buildings. The system employs OFDM at a centre frequency near 5.7 GHz, providing frequency-non-selective responses for each subcarrier. Wideband measurements were used to characterise the channel and to evaluate the expected system performance.

The height above traffic ensured that the channel response matrix, $\mathbf{H}$, had only small variations with time, attributed to the effects of wind on the tall buildings and antenna mounting structure. These temporal characteristics support the use of a closed-loop approach to signalling, in which the transmitter weight vectors are fed back from the receiver based on channel response estimates. The proposed signalling scheme was eigenbeamforming, in which the data substreams are weighted at the transmitter array such that they are emitted along the estimated right singular vectors, $\mathbf{v}_i$, and are weighted at the receiver array using vectors tuned to the left singular vectors, $\mathbf{u}_i$. If the channel estimates are ideal, the equivalent system model for the ith data substream is

$$\hat{s}_i = \mathbf{u}_i^H \mathbf{H} \mathbf{v}_i \mathbf{s} + \mathbf{u}_i^H \mathbf{n} = \quad_i s_i + \qquad (11)$$

where   is additive noise at the receiver output.

The sparsity of the scattering environment resulted in highly correlated spatial responses on the transmitter-receiver link pairs, and consequently there were only a small number of significant singular values of $\mathbf{H}$; in fact, only two eigenmodes supported transmission.

The SVD algorithm described above was applied to update the transmitter and receiver array weights, given by the dominant pair of right and left singular vectors, respectively. In the simulations shown here, the initial conditions were selected to be $\mathbf{V}(0) = \mathbf{U}(0) = \mathbf{I}$, and $\quad_i(0)$, $i = 1, \ldots, 8$ were randomly selected to have values in $(0, 1]$. The first 2000 transmitted vectors (0.04 s.) were used for training, and thereafter every $N_p = 10$th transmitted signal vector was assumed to be known at the receiver for updating the SVD estimate. These training symbol vectors were BPSK sequences with perfect autocorrelation properties [7]. For the training period, $= 0.999$, and during the subsequent data phase, $= 0.99$. The average ratio of received signal power to sensor noise power was 12 dB.

Fig. 1 shows the largest four true singular values (dotted lines) and the estimated singular values (solid lines) over the first two seconds of measured channel responses. While the magnitude is marginally underestimated, the trend is followed quite accurately.

When the estimated singular vectors, $\hat{\mathbf{u}}_i$ and $\hat{\mathbf{v}}_i$, suffer from errors, (11) degrades to

$$\hat{s}_i = \quad_i \hat{\mathbf{u}}_i^H \mathbf{u}_i \mathbf{v}_i^H \hat{\mathbf{v}}_i s_i + Z_i +$$

where the interference term resulting from the mismatch between the estimated and true singular vectors is, to a first-order approximation,

$$Z_i = \sum_{j \neq i} \left[\quad_i \hat{\mathbf{u}}_i^H \mathbf{u}_i \mathbf{v}_i^H \hat{\mathbf{v}}_j + \quad_j \hat{\mathbf{u}}_i^H \mathbf{u}_j \mathbf{v}_j^H \hat{\mathbf{v}}_j \right] s_j.$$

In Fig. 2, the absolute projections of the estimated left singular vectors onto the respective true singular vectors are shown for the two dominant modes; the performance for the right singular vectors is very similar. After initial adaptation, the separation between the modes is better than 15 dB. Because the channel parameters change slowly in this environment, it was seen in [7] that, even allowing for feedback delays, the update algorithm tracks the singular values and vectors with sufficient accuracy to achieve a threefold increase in data throughput using the eigenbeamforming technique relative to a system with single element antennas.

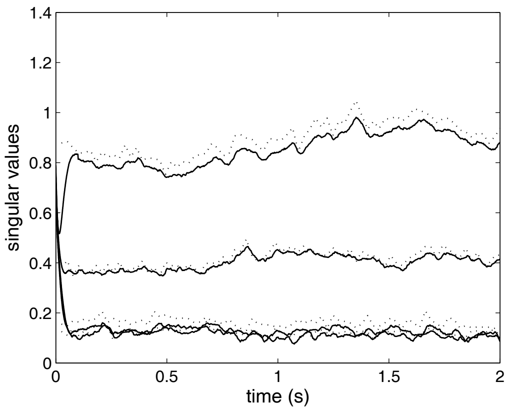

Figure 1: Estimated (solid lines) and true (dotted lines) singular values for eigenbeamforming application.

3.2 Adaptive V-BLAST

Another application for this SVD update algorithm is in the adaptive implementation of V-BLAST. V-BLAST is a successive interference cancellation scheme [8], in which zero-forcing weight vectors are formed at each step to null the interference from undetected signals, and detected signals are cancelled from the system equation. The signals are detected

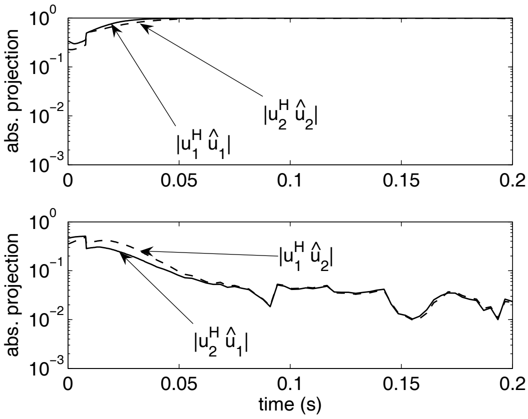

Figure 2: Projection of estimated singular vectors onto true ones for eigenbeamforming application.

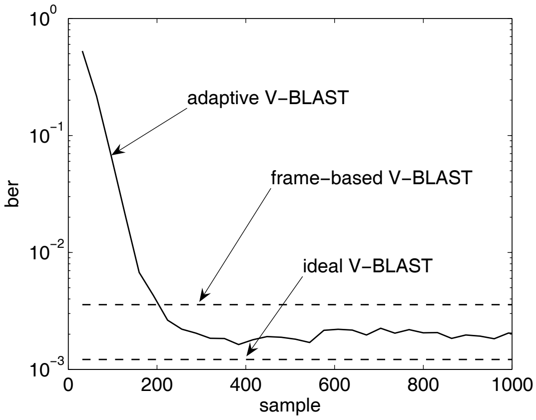

Figure 3: Bit error rate using efficient SVD update algorithm applied to adaptive V-BLAST for QPSK with $N = 4$.

in order of decreasing SNR, thereby minimising the propagation of detection errors. At step i, the undetected symbol which has the largest SNR is determined from the pseudo-inverse of $\mathbf{H}$ with $i-1$ columns, which correspond to previously detected and cancelled signals, zeroed. The pseudo-inverse of $\mathbf{H}$ is

$$\mathbf{H}^{\dagger} = \mathbf{V}\,\bar{}\,\mathbf{U}^{H}$$

where the diagonal elements of $\bar{}$ are given by

$$\bar{}_{i} = \begin{cases} 1/\ {}_{i} & {}_{i} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The V-BLAST algorithm therefore requires the pseudo-inverse, and hence the SVD, of matrix $\mathbf{H}$ with $i-1$ columns zeroed at steps $i = 1, \ldots, N$. It was proposed in [9] that, rather than estimating $\mathbf{H}$ and generating multiple SVDs each time, the SVD components of $\mathbf{H}$ and its subsets could be tracked. The details of the adaptive V-BLAST algorithm were presented in [9]; herein it is noted that the algorithm derived in Sec. 2 is suitable for updating the SVD components as new pilot symbols are received.

Bit error rate results obtained using the SVD update algorithm for an adaptive V-BLAST system using QPSK at an SNR of 12 dB with $N = 4$ are shown vs. the number of update samples in Fig. 3 for a fixed $\mathbf{H}$, starting from initial conditions $\mathbf{U}(0) = \mathbf{V}(0) = \mathbf{I}$ and ${}_{i}(0) = 1$, $i = 1, \ldots, N$. The mean performance, averaged over ten runs, using the frame-based V-BLAST is also shown. The training sequence in both cases was the same as used in Sec. 3.1. The performance of the adaptive V-BLAST using the efficient SVD update algorithm converges rapidly to close to the performance for a receiver with perfect channel knowledge, demonstrating that the estimated SVDs of each partial channel matrix converges, as required. The asymptotic performance is better than that obtained using the frame-based V-BLAST implementation, which does not use prior channel knowledge and has higher computational complexity.

4. CONCLUSIONS

An efficient SVD update algorithm has been presented which recursively computes left and right singular vectors and singular values of a transfer matrix $\mathbf{H}$. It has been shown to work well in a closed-loop MIMO fixed wireless access system using eigenbeamforming, where inaccurate SVDs lead to self-interference. It has also been used in conjunction with the V-BLAST algorithm to provide an adaptive implementation with significantly lower complexity and asymptotically better performance than the frame-by-frame version.

5. ACKNOWLEDGEMENTS

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