OPTIMIZATION OF NONUNIFORM PLANAR ARRAY GEOMETRY FOR DIRECTION OF ARRIVAL ESTIMATION

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ABSTRACT

In this work, we derived the probability of gross errors in direction of arrival (DOA) estimation for any arbitrary array under assumptions of two dimensional array geometry, isotropic array elements, deterministic incoming signal and AWGN. We proposed a metric function that can be used to optimize the array geometry, based on this derivation and available Cramér-Rao Bound (CRB) analysis in the literature. We used genetic algorithm as an optimization tool in our attempt to seek answer to the generic problem: *For a given two dimensional, bounded surface and maximum allowable probability of gross error at certain signal-to-noise ratio (SNR) value, what is the optimum geometry that maximizes the azimuth DOA estimation performance?*

1. INTRODUCTION

In most of the array signal processing applications, main interest is to obtain an accurate estimation of the Direction of Arrival (DOA) of a received signal by combining the observed samples of the signal at different sensor locations. Estimation accuracy of a given array depends upon certain characteristics of the array geometry as much as the estimation algorithm employed. Therefore, the path that leads to accurate estimation of DOA should start with correct localization of the array elements.

Given the number of array elements and the bounding surface, optimization of localization of the array elements can be based on two main criteria. The first criterion is Cramer-Rao Bound (CRB) which determines the maximum achievable estimation performance. CRB on estimation of DOA has been analyzed widely for different conditions and explicit formulas that relate the function to SNR and array sensor locations have already been derived [1]-[8]. Second criterion is selected as the probability of gross errors that may cause undesired ambiguous estimations. Definitely minimization of the probability of gross errors leads more accurate DOA estimates, especially in low SNR values.

Optimization of array geometry has received significant amount of attention among researchers [9]-[13]. However, very few of them takes the probability of gross errors into consideration [11] [12]. In this work, we start with this idea , further we state the relation between CRB and the probability of gross errors and extend the discussion to two dimensional arrays.

In Section 2, array manifold representation of two dimensional arrays based on [14] is given. In Section 3, received signal is modeled as a deterministic signal in AWGN. In Section 4, the probability of gross errors is expressed. The following section briefly states previously obtained results on CRB. In Section 6, we proposed a metric for optimization which simultaneously works on CRB for fine errors and probability of gross errors. In the last section we summarized the optimization procedure and presented the results.

2. ARRAY MANIFOLD REPRESENTATION OF TWO DIMENSIONAL ARRAYS

Geometry of a generic two dimensional array is given in Figure 1. In the figure locations of the sensors are represented by the position vector $\mathbf{p}$. Sensors are assumed to be isotropic. The azimuth angle

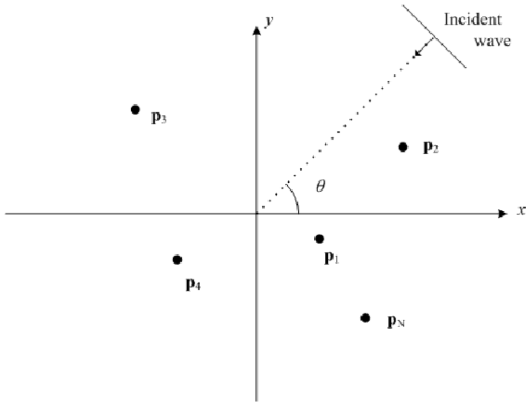

Figure 1: Two dimensional array geometry

of the incoming signal is given by θ Array sensors at the locations $p_n = [\; x_n \quad y_n \;]^T$ receive the incoming signals. Received signal at each sensor can be expressed as a delayed version of the incoming signal $g(t)$.

$$\mathbf{g}(t,\mathbf{p}) = \begin{bmatrix} g(t-\tau_1) \\ g(t-\tau_2) \\ \vdots \\ g(t-\tau_n) \end{bmatrix},$$

where delay is a function of the array sensor locations and the incidence angle $\tau_n = (-x_n \cos\theta - y_n \sin\theta)/c$.

If $g(t)$ is assumed to be a narrowband bandpass signal, it can be represented as,

$$g(t) = \mathrm{Re}\left\{\tilde{g}(t)e^{j\omega t}\right\}$$

and the signal at the nth sensor is,

$$g_n(t) \triangleq g(t-\tau_n) = \mathrm{Re}\left\{\tilde{g}(t-\tau_n)e^{j\omega(t-\tau_n)}\right\}.$$

The narrowband assumption implies that the phase and the amplitude of the signal is unchanged during the maximum propagation time across the sensors.

$$\tilde{g}(t-\tau_n) \cong \tilde{g}(t)$$

Therefore, the complex envelope of $g_n(t)$ will be

$$\tilde{g}_n(t) = \tilde{g}(t)e^{-j\omega\tau_n},$$

and

$$\tilde{\mathbf{g}}(t) = \begin{bmatrix} e^{j\frac{\omega}{c}(x_1\cos\theta + y_1\sin\theta)} \\ e^{j\frac{\omega}{c}(x_2\cos\theta + y_2\sin\theta)} \\ \vdots \\ e^{j\frac{\omega}{c}(x_n\cos\theta + y_n\sin\theta)} \end{bmatrix} \tilde{g}(t).$$

As a result, for any arbitrary N-element array, array manifold is given by

$$\mathbf{A}(\theta)=\begin{bmatrix} e^{j\frac{\omega}{c}(x_1\cos\theta+y_1\sin\theta)} \\ e^{j\frac{\omega}{c}(x_2\cos\theta+y_2\sin\theta)} \\ \vdots \\ e^{j\frac{\omega}{c}(x_n\cos\theta+y_n\sin\theta)} \end{bmatrix}. \qquad (1)$$

3. MODELING OF THE RECEIVED SIGNAL

Received signal vector of the array is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} g(t-\tau_1) \\ g(t-\tau_2) \\ \vdots \\ g(t-\tau_n) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \\ \vdots \\ n_n(t) \end{bmatrix}.$$

The signal $g(t)$ is assumed to be a known bandpass narrowband signal and the noise $n(t)$ is assumed to be white Gaussian process. Noise is spatially uncorrelated between sensors.

After coherent demodulation and Nyquist rate sampling the model can be reduced to the following [14],[8] and usually referred to as narrowband time domain snapshot model.

$$\mathbf{x}(k)=\mathbf{A}(\theta)g(k)+\mathbf{n}(k) \qquad k=1,2,\cdots,K$$

$\mathbf{n}(k)$'s are independent, zero mean Gaussian random variables.

$$\begin{aligned} E\{\mathbf{n}(k)\} &= 0 \\ E\left\{\mathbf{n}(k)\mathbf{n}^H(k)\right\} &= \sigma^2\mathbf{I} \\ E\left\{\mathbf{n}(k)\mathbf{n}^H(l)\right\} &= 0 \end{aligned}$$

Under the assumptions stated, $\mathbf{x}(k)$ is Gaussian distributed with the following properties,

$$\begin{aligned} E\{\mathbf{x}(k)\} &= \mathbf{A}(\theta)g(k) \\ Var\{\mathbf{x}(k)\} &= \sigma^2\mathbf{I} \end{aligned}$$

Note that, σ^2 is the noise sample power in receivers bandwidth of B and equals to $2N_0B$.

4. GROSS ERRORS

Probability density of one snapshot will be

$$p(\mathbf{x}(k)\mid\theta)=\frac{1}{\pi^N\sigma^{2N}}\exp(-\frac{1}{\sigma^2}\left\|\mathbf{x}(k)-\mathbf{A}(\theta)g(k)\right\|^2).$$

Since the snapshots are statistically independent, joint pdf of snapshots are obtained as,

$$p(\mathbf{x}\mid\theta)=\prod_{k=1}^{K}\frac{1}{\pi^N\sigma^{2N}}\exp(-\frac{1}{\sigma^2}\left\|\mathbf{x}(k)-\mathbf{A}(\theta)g(k)\right\|^2).$$

The log-likelihood function is

$$\ln p(\mathbf{x}\mid\theta)=C-\frac{1}{\sigma^2}\sum_{k=1}^{K}\left\|\mathbf{x}(k)-\mathbf{A}(\theta)g(k)\right\|^2.$$

Let us suppose that there are two candidate angles of arrival θ_1 and θ_2. In hypothesis one, namely H_1, direction of arrival is θ_1. Similarly, in hypothesis two (H_2), direction of arrival is θ_2 . Based on the observation of snapshots we will decide whether H_1 or H_2 is true.

$$\begin{aligned} H_1 &: \quad \mathbf{x}(k)=\mathbf{A}(\theta_1)g(k)+\mathbf{n}(k) \\ H_2 &: \quad \mathbf{x}(k)=\mathbf{A}(\theta_2)g(k)+\mathbf{n}(k) \end{aligned}$$

The likelihood ratio test which minimizes the probability of the wrong hypothesis decision (gross error) is

$$\psi\triangleq\ln p(\mathbf{x}\mid\theta_1)-\ln p(\mathbf{x}\mid\theta_2)\underset{H_1}{\overset{H_2}{\lessgtr}}0.$$

When H_1 is true, ψ becomes

$$\begin{aligned} \psi &= \frac{1}{\sigma^2}\sum_{k=1}^{K}[g^*(k)[\mathbf{A}^H(\theta_1)-\mathbf{A}^H(\theta_2)]+\mathbf{n}^H(k)] \\ &\quad [[\mathbf{A}(\theta_1)-\mathbf{A}(\theta_2)]g(k)+\mathbf{n}(k)]-\mathbf{n}^H(k)\mathbf{n}(k). \end{aligned}$$

ψ is the sum of independent random numbers. Thus, for large number of samples, the distribution of ψ is approximately normal.

$$\begin{aligned} \mu &\triangleq E\{\psi\mid H_1\} \\ &= \frac{2BE}{\sigma^2}\left\|\mathbf{A}(\theta_1)-\mathbf{A}(\theta_2)\right\|^2, \end{aligned} \qquad (2)$$

where

$$E=\frac{1}{2B}\sum_{k=1}^{K}g(k)g^*(k)$$

is the energy of the received bandpass signal on any one of the sensors. Here, a convenient definition of SNR is the signal energy to noise power spectral density. That is, $SNR=E/N_0$. Note that, μ can be seen to be as the SNR times the square of the distance between the two array manifold vectors corresponding the two different incidence angles.

Similarly,

$$E\left\{\psi^2\mid H_1\right\}=\mu^2+2\mu.$$

Therefore,

$$Var\{\psi\mid H_1\}=2\mu.$$

Finally, the probability of gross error is obtained by,

$$P_g=\text{erfc}(\sqrt{\frac{\mu}{2}}), \qquad (3)$$

where erfc is the complementary error function.

5. CRAMÉR-RAO BOUND FOR ESTIMATION OF DIRECTION OF ARRIVAL

Cramér-Rao bounds on direction of arrival estimation have been thoroughly analyzed in literature. By using the previous results in references [1]-[8], CR bound on DOA estimation can easily be derived for the model given in Section 3:

$$CRB(\theta)=\left[\frac{4BE}{\sigma^2}\frac{\omega^2}{c^2}\sum_{i=1}^{N}[(y_i-y_c)\cos\theta-(x_i-x_c)\sin\theta]^2\right]^{-1},$$

where, x_c and y_c are the coordinates of the center of the array given by

$$x_c=\frac{1}{N}\sum_{i=1}^{N}x_i, \qquad y_c=\frac{1}{N}\sum_{i=1}^{N}y_i. \qquad (4)$$

6. PROPOSED METRIC

Let us now consider the function $\left\|\mathbf{A}(\theta_1)-\mathbf{A}(\theta_2)\right\|^2$ that appears explicitly in (2). In references [11] and [12], this particular function is offered as a measure of similarity between two steering vectors of each incidence angle. As (3) shows, it is obvious that the higher the value of the function, the lesser the probability of gross errors. By tracing the following analysis we will show that the term is also related to the CRB.

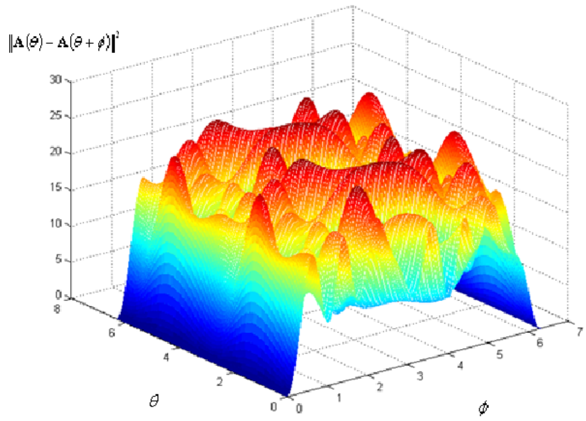

Figure 2: Sample sketch of function $\|\mathbf{A}(\theta)-\mathbf{A}(\theta+\phi)\|^2$

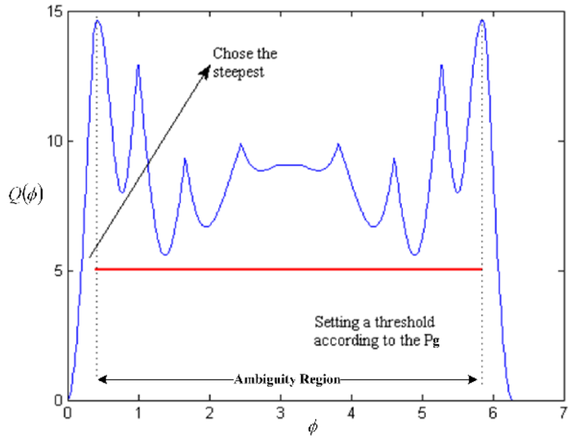

Figure 3: Sketch of function $Q(\phi)$. Optimization wrt P_g and CRB is shown

Without loss of the validity of derivation given in Section 4, we can take the center of array given in (4), as the reference point for the array. Hence, (1) can be rewritten as,

$$\mathbf{A}(\theta) \triangleq \begin{bmatrix} e^{j\frac{\omega}{c}((x_1-x_c)\cos\theta+(y_1-y_c)\sin\theta)} \\ e^{j\frac{\omega}{c}((x_2-x_c)\cos\theta+(y_2-y_c)\sin\theta)} \\ \vdots \\ e^{j\frac{\omega}{c}((x_n-x_c)\cos\theta+(y_n-x_c)\sin\theta)} \end{bmatrix}.$$

Thus, we can obtain the followings.

$$\|\mathbf{A}(\theta)-\mathbf{A}(\theta+\phi)\|^2 = 2N - \mathrm{Re}\left\{\mathbf{A}^H(\theta)\mathbf{A}(\theta+\phi)\right\} \quad (5)$$
$$= 2N - 2\sum_{i=1}^{N}\cos\left(\frac{\omega}{c}((x_i-x_c)(\cos(\theta+\phi)-\cos\theta)\right.$$
$$\left.+(y_i-y_c)(\sin(\theta+\phi)-\sin\theta))\right).$$

The term in summation in (5) has a Taylor series expansion near $\phi=0$ as,

$$1-\frac{\phi^2}{2}\frac{\omega^2}{c^2}[(y_i-y_c)\cos\theta-(x_i-x_c)\sin\theta]^2+\mathrm{H.O.T.} \quad (6)$$

By using (6) in (5), we obtain $\|\mathbf{A}(\theta)-\mathbf{A}(\theta+\phi)\|^2$ around $\phi=0$ approximated as,

$$\|\mathbf{A}(\theta)-\mathbf{A}(\theta+\phi)\|^2 \simeq$$
$$\frac{\phi^2}{2}\frac{\omega^2}{c^2}\sum_{i=1}^{N}[(y_i-y_c)\cos\theta-(x_i-x_c)\sin\theta]^2.$$

Hence, as function $\|\mathbf{A}(\theta)-\mathbf{A}(\theta+\phi)\|^2$ gets high around $\phi=0$, CRB gets low. Figure 2 gives a sample sketch of the function.

Obviously, when $\phi=0$ the function is zero for all values of θ. Steepness of the function around $\phi=0$ provides less estimation error around the true bearing of the signal. In other words, CRB on DOA estimation gets lower as the steepness increases. Obviously, the function evaluating to zero along ϕ axis except $\phi=0$ and 2π indicates totally ambiguous estimations and getting close to zero means probability of gross errors becomes higher.

Based on the discussion above we define the following metric function:

$$Q(\phi) \triangleq \min_{\theta}\left[\|\mathbf{A}(\theta)-\mathbf{A}(\theta+\phi)\|^2\right]. \quad (7)$$

- Maximization of $Q(\phi)$ around $\phi=0$ indicates a lower CRB based on the discussion in Section 5 and leads to a better fine error.
- Maximization of $Q(\phi)$ elsewhere indicates a lower probability of gross error.

7. OPTIMIZATION BY USING GENETIC ALGORITHM

We first define the bounded region that the sensors can be located. Optimization starts with the setting of maximum allowable gross error probability for a certain SNR. By using (3) and (2), we can find the minimum value of $Q(\phi)$ satisfying this condition.

After setting this threshold, we should find the optimum array geometry that has the steepest rising edge around $\phi=0$. Figure 3 gives the visualization of the procedure. Ambiguity region is defined as the region outside the first peaks of $Q(\phi)$ in both directions.

We applied genetic algorithm to the problem. Genetic algorithm starts with a sample population. All population members are created randomly restricted to the bounded area specified. Array elements are not allowed to be placed closer than 0.1λ. Candidates are filtered by comparing their P_g with respect to a threshold. Each population member is assigned a fitness value associated with the CRB. After crossover, offsprings are filtered again and added to the population. Process is repeated for a certain number iterations.

As an example, let us design an array that all array elements should be in a circle that has a radius of 2λ and has the maximum probability of gross error 10^{-5} at 5 dB SNR. The results obtained with genetic algorithm is compared with the default uniform circular array (UCA). Figure 4 shows the optimized array geometry and the UCA. Dashed line shows the bounds of the allowed space to place the sensors. Figure 5 shows the function $Q(\phi)$ for each array geometry. It is clearly seen that UCA does not satisfy the given P_g specification. If we take a close look at Figure 5 around $\phi=0$, we can see that the DOA estimation variance for fine errors of UCA is slightly better than the nonuniform array. This can be seen in Figure 6 where achievable CRB of both geometries are sketched for all values of incidence angle θ.

Gross error risk is significantly reduced with a small compromise of DOA estimation variance . The algorithm can be applied to any arbitrary two dimensional region.

8. CONCLUSIONS

We proposed an optimization procedure in order to improve fine error variance under the condition of a given probability of gross errors. These two parameters are inherently contradicting and op-

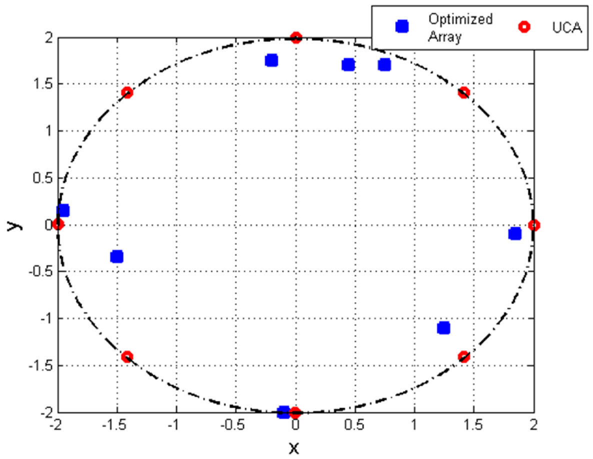

Figure 4: Array geometries for both UCA and optimized array

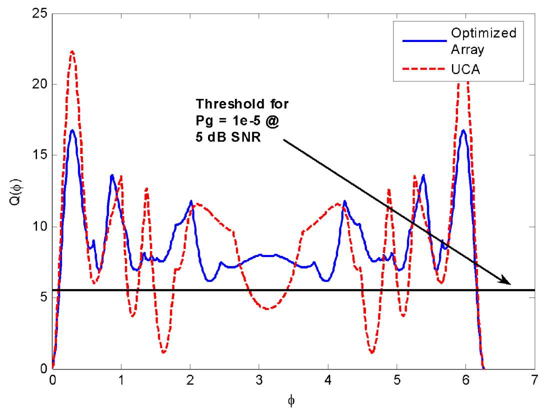

Figure 5: $Q(\phi)$ of both UCA and optimized array

timization of one of them does not necessarily lead to a good array design.The proposed method outperforms especially when high precision DOA estimates are needed while the number of sensors are limited and the allowed region is large. Under these conditions, using a uniform planar array leads to intolerable gross errors. The problem may easily be reversed. Setting a DOA estimation variance and optimizing for the best achievable gross error probability is an another possible option with this method.

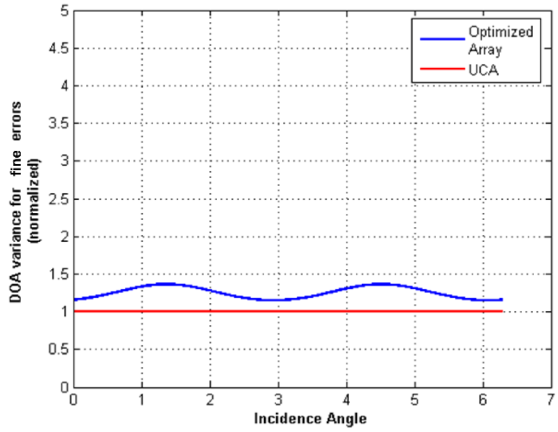

Figure 6: DOA estimation variance for fine errors of both optimized array and UCA

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