ORDER STATISTICS-BASED UNBIASED HOMOMORPHIC SYSTEM TO REDUCE MULTIPLICATIVE NOISE

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ABSTRACT

In this paper, we propose an order statistics-based unbiased homomorphic system to reduce multiplicative noise. The design of such a system is based on the probability density function (PDF) of the noise. First, we generalize the order statistics-based nonlinear filter called the sampled function weighted order (SFWO) filter proposed in [1] to reduce additive noise, to the case when the additive noise is not symmetric. This generalized SFWO (GSFWO) filter is then used in a homomorphic system to reduce multiplicative noise corrupting a signal. It is shown that the output from this GSFWO filter-based homomorphic system will be biased and hence, a bias compensation technique is applied to the output to get the unbiased estimate. A study of the qualitative and quantitative performance of the proposed GSFWO filter-based unbiased homomorphic system in reducing multiplicative noise is carried out and compared to that of some of the existing ones. It is found that the proposed GSFWO filter-based system consistently outperforms the others irrespective of the type of the PDF of the multiplicative noise.

1. INTRODUCTION

Signal corrupted by noise of a multiplicative nature is often encountered in many applications such as coherent imaging systems and nonlinear communication channels. Such a corruption is modelled as

$$ = \times \qquad (1)$$

where represents the corrupted signal, represents the original uncorrupted signal and the multiplicative noise. Kuan et al. in [2] proposed a filter to reduce multiplicative noise by reframing the noise model given by (1) into a suitable additive form and then minimizing the mean square error (MSE) between the desired response and the actual output. Sample mean, sample median or edge adaptive Wiener filter-based homomorphic systems have also been used to reduce multiplicative noise [3]. The design of the above mentioned filters does not consider any characterisitic of the noise. As in most applications, the probability density function (PDF) of the noise is known or can easily be determined for the reduction of the multiplicative noise, it would be desirable to have a system whose design is based on the PDF of the noise.

In this paper, we propose an order statistics-based unbiased homomorphic system (see Figure 1), wherein an order statistic filter is used within an unbiased homomorphic system to reduce the multiplicative noise. The coefficients of the order statistic filter are obtained from the PDF of the multiplicative noise.

To reduce the multiplicative noise, the proposed homomorphic system uses the natural logarithm to transform the multiplicative nature of corruption into an additive one and then processes the resulting corrupted signal using a filter to reduce the additive white noise. The sampled function weighted order (SFWO) filter proposed in [1], whose design is easy and is based on the PDF of the noise to the filter, could be used as the filter to reduce the additive noise provided the noise has symmetric PDF. Unfortunately, the additive noise obtained after the natural logarithm operation within the homomorphic system might not have a symmetric distribution.

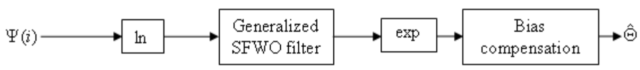


Figure 1: The proposed GSFWO filter-based homomorphic system to reduce multiplicative noise

A generalization of the SFWO filter is proposed in this paper relaxing the symmetric PDF condition of the additive noise. The resulting filter is refered to as the generalized SFWO (GSFWO) filter and is used to reduce the existing additive nature of corruption after the natural logarithmic transformation within the system. An exponential function is then applied to the output of this filter. It is shown that the output after the exponentiation would be biased and hence, a bias compensation technique is applied to the output to get the unbiased estimate. The expressions required to determine the coefficients of the GSFWO filter from the PDF of the multiplicative noise are given in Section 3.

2. GENERALIZATION OF THE SFWO FILTER

In this section, a generalization of the SFWO filter to reduce an additive white noise is proposed by relaxing the symmetry condition of the noise PDF. This design extends the usability of the filter to applications where additive noise with asymmetric PDF is encountered, e.g., a homomorphic system. As in [1], the classical problem of estimating a constant amplitude signal from the additively corrupted observed samples (i) within a filter window is considered:

$$(i) = \quad + n(i) \qquad (2)$$

where $n(i)$ is assumed to be a stationary, white, zero-mean noise. It is also assumed that the uncorrupted signal and the noise n are uncorrelated to each other. However, no assumption about the shape of the noise PDF is made, unlike the design given in [1]. Let the output of the GSFWO filter be given as

$$(i) = \frac{\sum_{j=1}^{r} h_j \quad_{(j)}(i)}{\sum_{j=1}^{r} h_j} \tag{3}$$

where r is the number of elements within a filter window, $_{(j)}$ stands for the j^{th} largest element within the filter window and h_j $(j = 1, 2, ..., r)$ stands for the unnormalized coefficients of the GSFWO filter, which will be determined using this design. The result of minimization of the MSE between the desired response and the actual response will yield the solution:

$$h_j = \sum_{k=1}^{r} \bar{C}_{jk} \tag{4}$$

where C_{jk} are the elements of the covariance matrix of v, v being a unit variance random variable related to the additive noise in the form $n = \quad v$. is the standard deviation of the noise n and is always positive and non-zero. $\bar{C}_{jk}$ stands for the elements of the inverse of the covariance matrix.

Let the standard PDF of v be $f_v(v)$ and the cumulative distributive function (CDF) of v be $F_v(v)$. Now, to obtain the values of the coefficients given by (4) their asymptotic behaviour ($r \rightarrow \quad$) shall be examined. As presented in [4], the samples of v, $v_{(j)}$ and $v_{(k)}$, are asymptotically distributed (as $r \rightarrow \quad$) according to the normal bivariate distribution with the covariance:

$$C_{jk} = \frac{_j(1 - \quad_k)}{r f_v(v_j) f_v(v_k)},\ 1 \leq i \leq j \leq r \tag{5}$$

where

$$_j = \frac{j}{r+1} \text{ and } v_j = F_v^{-1}(\quad_j) \tag{6}$$

It is assumed in (5) that $f_v(v)$ is nonzero. It is also assumed that f_v' and f_v'' exist for $F_v^{-1}(0) < v < F_v^{-1}(1)$. As the covariance matrix is singular in nature, the well known Moore Penrose equations are used to find the elements of its pseudo-inverse and the matrix thus obtained is considered as the inverse of the covariance matrix. The elements of the inverse of the covariance matrix may be derived as

$$\begin{aligned}
\bar{C}_{jj} &= \frac{2 r f_v^2(v_j)}{(1 - j \quad)},\ 1 \leq j \leq r-1 \\
\bar{C}_{jk} &= \frac{-r f_v(v_j) f_v(v_k)}{(1 - j \quad)},\ j, k = 1, 2, 3 ... r-1, |\ j - k\ | \\
\bar{C}_{rr} &= \frac{r f_v^2(v_r)}{(1 - r \quad)} \\
\bar{C}_{jk} &\approx 0,\ \text{ for } |\ j - k\ | > 1
\end{aligned} \tag{7}$$

where

$$= \frac{1}{r+1} \tag{8}$$

Using the expression for the inverse of the covariance matrix given by (7) in (4), and after some algebraic manupulations, we obtain the coefficients given by

$$h_j = \ _v''\left(F_v^{-1}\left(\frac{j}{r+1}\right)\right),\ j = 1, 2, 3 ... r \tag{9}$$

where $_v(v) = \ln(f_v(v))$.

3. COEFFICIENTS OF THE GSFWO FILTER WITHIN THE HOMOMORPHIC SYSTEM

The first step in the proposed homomorphic system is to take the natural logarithmic transform of the observed corrupted signal. Hence, applying natural logarithm to both sides of (1), which gives the multiplicative noise model, we get

$$\ln \quad = \ln \quad + \ln \tag{10}$$

Inserting an index i, (10) can be written as

$$(i) = \quad + n(i) \tag{11}$$

where n is a zero-mean noise, $= \ln$ and $= \ln \quad + m$, m being the mean of ln . We assume the multiplicative noise to be stationary, white and uncorrelated to the signal , in which case (11) becomes exactly the same as (2) and the GSFWO filter can be applied. Now, the coefficients of the GSFWO filter are obtained considering the relation between the random variables v and given by $v = ln(\quad)$. Let $f\ (\quad)$ and $F\ (\quad)$ be the standard PDF and the CDF of the multiplicative noise , respectively. From the relation between the random variables v and , we get the relationship between the corresponding inverse CDFs as

$$F_v^{-1} = \ln F^{-1} \tag{12}$$

whereas their corresponding PDFs are related by

$$f_v(v) = f\ (\exp(v))\ |\ J\ | \tag{13}$$

where J is the Jacobian of the transformation $v = ln(\quad)$ [5]. From (13) and the relation $(\quad) = \ln(f\ (\quad))$, we obtain the relations:

$$_v'(v) = 1 + \ '\left(\exp(v)\right) \cdot \exp(v) \tag{14}$$

$$\begin{aligned}
_v''(v) = \ &'\left(\exp(v)\right) \cdot \exp(v) + \\
&''\left(\exp(v)\right) \cdot \exp(2v)
\end{aligned} \tag{15}$$

Using (9), (12), (14) and (15), the coefficients of the GSFWO filter within the proposed system can be obtained, when the PDF $f\ (\quad)$ of the multiplicative noise is known.

4. BIAS COMPENSATION

In this section, it is shown that the presence of m makes the output after the exponentiation biased and hence a corresponding bias compensation procedure is proposed. Let the output from the GSFWO filter within the homomorphic system be denoted by $\hat{}$, which is given by

$$\hat{} = \ln(\hat{}) + \hat{m} \tag{16}$$

where $\hat{}$ is the unbiased estimate of the original uncorrupted signal and $\hat{m}$ is the estimate of the shift. Now, taking the exponential of the estimate $\hat{}$, we get the biased estimate which is expressed as

$$\hat{}\,' = \exp \hat{} = \exp\left(\ln(\hat{}) + \hat{m}\right) = \hat{} + \exp(\hat{m}) \tag{17}$$

As can be seen, the output is biased by a factor $\exp(\hat{m})$. This bias can be compensated as follows. Taking the expected value on both sides of (17),

$$E[\hat{}\,'] = E[\hat{}] \cdot \exp(\hat{m}) \tag{18}$$

Now by consistency theory of estimates [6]

$$E[\ ] = E[\hat{\ }] \tag{19}$$

and since   is assumed to have unit mean, we have

$$E[\ ] = E[\ ] \tag{20}$$

Using (18), (19) and (20), a bias compensation constant   is defined and the compensation is achieved as follows

$$= \exp(m) = \frac{E[\hat{\ }']}{E[\ ]} \tag{21}$$

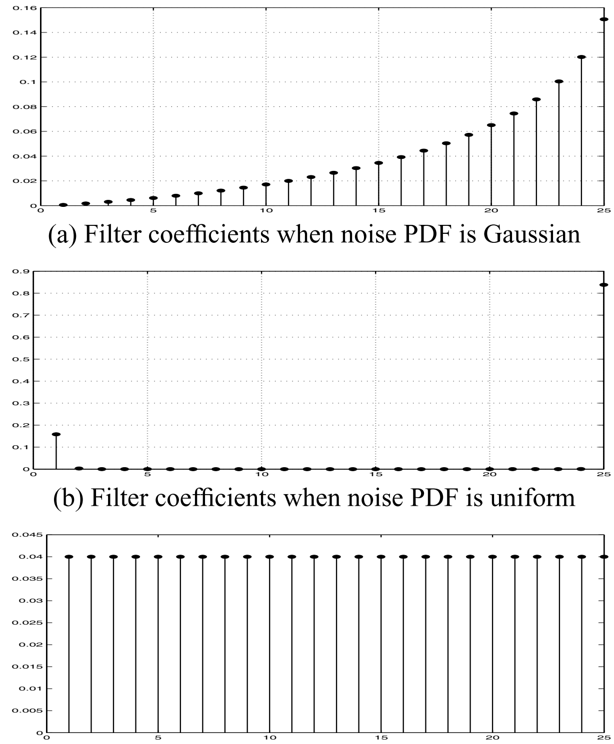
$$\hat{\ } = \frac{\hat{\ }'}{\ } \tag{22}$$

Thus, as both the biased recovered signal ($\hat{\ }'$) and the observed signal ( ) are available, the unbiased estimate of the original signal ($\hat{\ }$) is obtained by using (22).

5. SIMULATIONS

In this section, the performance of the proposed GSFWO filter-based unbiased homomorphic system (UH-GSFWOF) to reduce a multiplicative noise is compared to that of a few known filters like the Kuan et al. filter (KF), the sample mean filter-based homomorphic system (H-SMF), the sample median filter-based homomorphic system (H-SMDF) and the edge-adaptive Wiener filter-based homomorphic system (H-EAWF). Next, we use the proposed bias compensation technique in the H-SMF, the H-SMDF and the H-EAWF to get the sample mean filter-based unbiased homomorphic system (UH-SMF), the sample median filter-based unbiased homomorphic system (UH-SMDF) and the edge-adaptive Wiener filter-based unbiased homomorphic system (UH-EAWF) respectively, and give the corresponding results. Three different types of multiplicative noise of variance 0.25 (normalized with respect to maximum greyscale value: 255) having Gaussian, uniform and lognormal PDFs are considered, and the corresponding normalized coefficients of the GSFWO filter within the system are shown in Figure 2. Two standard images, namely, the Pepper and Goldhill images are considered to analyze the performance.

Quantative performance of the filters are compared using the MSE as the quantitative measure, which is calculated between the desired response and the actual response, whereas in case of noisy image it is calculated between the original image and the noisy image. The results are tabulated in Tables 1 and 2, where it is evident that the proposed system outperforms the others in reducing multiplicative noise having different distributions. This better performance of the proposed GSFWO filter-based unbiased homomorphic system can be attributed to the fact that the coefficients of the filter are derived based on the type of distribution of the noise. It can be observed from the tables that the UH-SMF gives the same result as that of the proposed system when the noise is lognormally distributed. This should be the case since the GSFWO filter in the proposed system reduces to a sample mean filter for a multiplicative noise with lognormal distribution (see Figure 2). However, for other types of noise considered, the proposed system gives a better performance. It can also be observed from Tables 1 and 2 that the UH-EAWF



(a) Filter coefficients when noise PDF is Gaussian

(b) Filter coefficients when noise PDF is uniform

(c) Filter coefficients when noise PDF is lognormal

Figure 2: Coefficients of the GSFWO filter within the homomorphic system for multiplicative noise with different distributions (5x5 filter window is considered)

gives equally good or better results than the proposed system when the noise has a lognormal distribution, whereas the later outperforms the former in other cases. This is due to the fact that the Wiener filter used is adapted to the edges, and hence does not blur the edges. It is also evident from the tables that the systems with the bias compensation give better results compared to the those of the corresponding one without the bias compension. This clearly points out the need for bias compensation.

Figure 3 shows the qualitative performance of the various filters considered in reducing multiplicative noise. Multiplicative noise having Gaussian PDF is alone considered here due to space constraint. From this figure, it can be seen that the proposed unbiased homomorphic system effectively reduces the multiplicative noise and gives the best results amongst all the filters considered.

6. CONCLUSION

In this paper, we have generalized the SFWO filter introduced in [1] to the case when the additive noise corrupting the signal is not symmetric. This generalized SFWO (GSFWO) filter is then used in an unbiased homomorphic system to reduce multiplicative noise corrupting a signal. A study of the qualitative and quantitative performance of the proposed GSFWO filter-based unbiased homomorphic system in reducing multiplicative noise has been carried out using two standard images, namely, the Pepper and Goldhill, and compared to that of some of the existing ones. It has been found that the proposed GSFWO filter-based system consistently

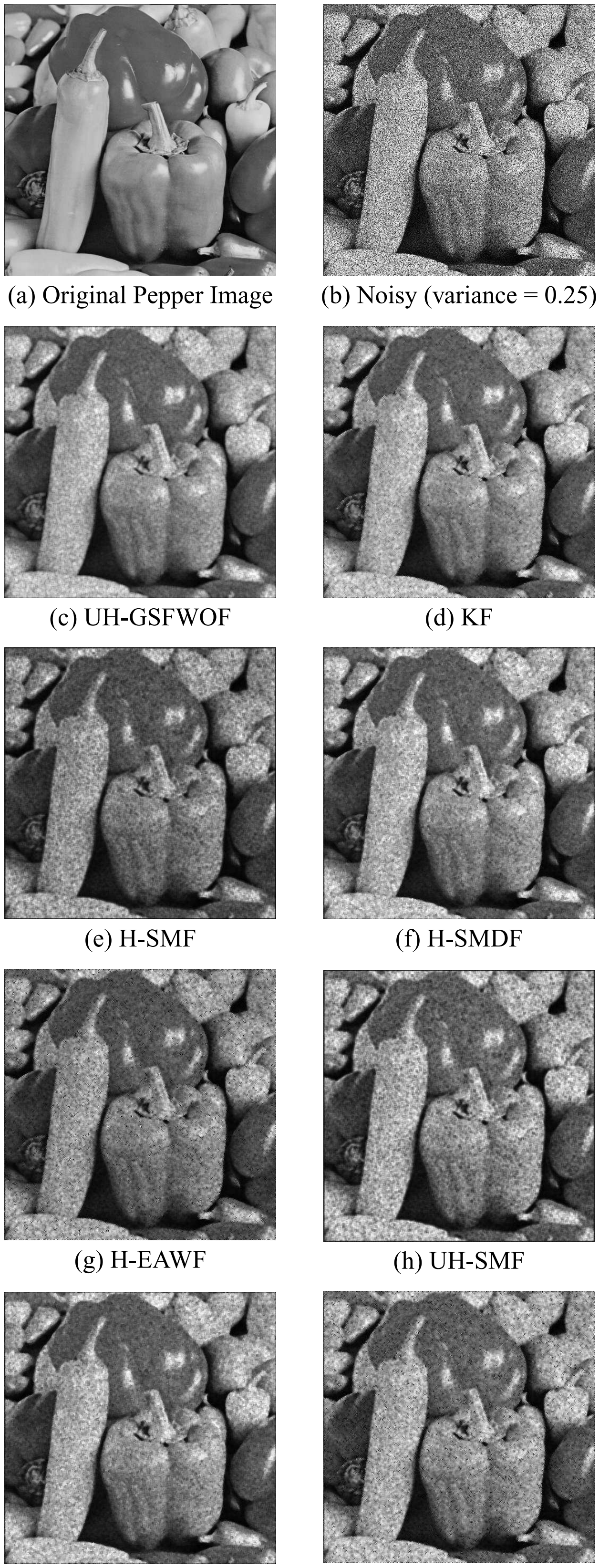
MSE	Gaussian noise PDF	Uniform noise PDF	Lognormal noise PDF
Noisy	3159.8	3604.8	1917.1
proposed UH-GSFWOF	226.54	289.4	326.67
KF	295.7	329.17	762.4
H-SMF	859.52	848.71	431.29
H-SMDF	371.37	596.43	526.61
H-EAWF	790.72	724.76	328.03
UH-SMF	582.81	524.91	326.67
UH-SMDF	371.75	578.49	342.18
UH-EAWF	605.47	465.39	213.6

Table 1: MSE for the various filters in reducing multiplicative noise using the 'Pepper' image

MSE	Gaussian noise PDF	Uniform noise PDF	Lognormal noise PDF
Noisy	2432.8	2739.3	1595.1
proposed UH-GSFWOF	188.97	217.57	193.87
KF	248.01	267.09	607.95
H-SMF	598.13	591.28	275.07
H-SMDF	295.37	443.7	424.21
H-EAWF	634.67	576.82	284.27
UH-SMF	406.15	367.1	193.87
UH-SMDF	280.25	407.62	209.04
UH-EAWF	512.89	410.35	199.18

Table 2: MSE for the various filters in reducing multiplicative noise using the 'Goldhill' image

outperforms the others, irrespective of the type of distribution of the multiplicative noise.



(a) Original Pepper Image (b) Noisy (variance = 0.25)

(c) UH-GSFWOF (d) KF

(e) H-SMF (f) H-SMDF

(g) H-EAWF (h) UH-SMF

(i) UH-SMDF (j) UH-EAWF

Figure 3: Qualitative results using the Pepper image showing the performance of the various filters in reducing multiplicative noise having Gaussian distribution

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